213 Final Exam Review Part 1 Solutions

December 8, 2013

1.

(a) $f'(x) = 2xe^x + x^2e^x = e^x(x^2+2x)$;

(b) f is a constant so $f' = 0$;

(c) $f'(x) = \frac{(x^3+1)(2x+2)-(x^2+2x+3)(3x^2)}{(x^3+1)^2}$;

(d) $f'(x) = (2x+4)e^{x^2+4x+1}$;

(e) $f'(x) = \frac{(x^3+1)(2e^{x^3}+2x(3x^2e^{x^3}))-2xe^{x^3}(3x^2)}{(x^3+1)^3} = \frac{e^{x^3}(x^3+1)(2+6x^3)-6x^3e^{x^3}}{(x^3+1)^3}$;

(f) $f'(x) = e^xe^{e^x}$;

(g) $f'(x) = \frac{2x+e^{x+1}}{x^2+e^{x+1}}$.

2.

(a) $f'(0) = e^0(0+0) = 0$;

(b) $f'(0) = 0$;

(c) $f'(0) = \frac{(0+1)(0+2)-(0+0+3)(0)}{(0+1)^3} = 2$;

(d) $f'(0) = 4e^1 = 4e$;

(e) $f'(0) = \frac{e^0(0+1)(0+2)-6(0)e^0}{(0+1)^3} = 2$;

(f) $f'(0) = e^0e^{e^0} = e$;

(g) $f'(0) = \frac{0+e^1}{0+e^1} = 1$;

3. For both of these we do the analysis, and leave the actual graphing to the reader.

(a) To compute critical points we first take the derivative of f as $f'(x) = 3x^2 - 3$. Thus critical points occur when $f'(x) = 3x^2 - 3 = 0 \implies x = \pm 1$. To classify these critical points we must use the first derivative test. A test value to the left of negative 1 is $f'(-2) = 12 - 3 = 9 > 0$, between -1 and 1 is $f'(0) = -3 < 0$, and to the right of 1 is $f'(2) = 12 - 3 = 9$. We conclude that the function is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$. Moreover, we can also conclude that $(-1, f(-1)) = (-1, 3)$ is a max while $(1, f(1)) = (1, -1)$ is a min.

For inflection points we compute the second derivative as $f''(x) = 6x$, and so the only inflection point is at $x = 0$.

Finally, because the leading term of f has an odd exponent, we know that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x)$.

(b) Once again we compute our derivative as $f'(x) = (x+1)e^x$ using the product rule. Critical points will therefore occur when $(x+1)e^x = 0$ and, because the exponential is always positive, this is when $x = -1$. We see that $f'(0) = 1e^1 = e > 0$, while $f'(-2) = -1e^{-2} = -e^{-2} < 0$. It follows that the function is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$, which then implies that

$(-1, f(-1)) = (-1, -e^{-1})$ is a minimum.

For inflection points we compute the second derivative using the product rule as $f''(x) = (x+2)e^x$. The only inflection point is therefore $(-2, -2e^{-2})$.

Finally, we immediately notice that $\lim_{x\to\infty} xe^x = \infty$. This is because both x and e^x get very large as x gets very large. What is less obvious is that $\lim_{x\to-\infty} xe^x = 0$. In particular, while x is going to negative infinity, e^x is going to zero at a rate which is faster. This is a fact from 211.

4. By definition, a rectangle is circumscribed in a circle if and only if the diagonal of the rectangle as has the same length as the diameter of the circle. In our particular case that tells us that the diagonal of the rectangle must be 1. From basic geometry we know that the length of the diagonal is precisely $D = 1 = \sqrt{l^2 + w^2}$, where l is the length of the rectangle, and w is the width. Solving for w we find that $\sqrt{1 - l^2} = w$, and so plugging this into the area formula $A = lw = l\sqrt{1 - l^2}$. We want to find the maximum value of A and so we look for critical points of the function $l\sqrt{1 - l^2}$. The derivative is,

$$A' = \sqrt{1 - l^2} - \frac{l^2}{\sqrt{1 - l^2}} = \frac{1 - l^2 - l^2}{\sqrt{1 - l^2}} = \frac{1 - 2l^2}{\sqrt{1 - l^2}},$$

and this is zero when $l = \frac{\pm 1}{\sqrt{2}}$. We ignore the negative solution, and find that the maximum area is $A = \frac{1}{2}$.

5. Compute the following indefinite integrals:

   (a) $\int (x^2 + 3x + 1)dx = \frac{1}{3}x^3 + \frac{3}{2}x^2 + x$;

   (b) We apply the substitution $u = x + 1$ so that $du = dx$ and our integral becomes $\int \frac{u-1}{u} du = \int 1 - \frac{1}{u} du = u + \ln|u| + c = x + 1 + \ln|x + 1| + c$;

   (c) The substitution is $u = x^2 + 1$ so that $du = 2xdx$ and our integral becomes $\int \frac{du}{u} = \ln|u| + c = \ln|x^2 + 1| + c$;

   (d) We integrate by parts. Set $u = \ln(x)$ and $dv = x^2 dx$ so that $du = \frac{1}{x}$ and $v = \frac{1}{3}x^3$. Thus,

$$\int x^2 \ln(x)dx = \frac{1}{3}x^3 \ln(x) - \int \frac{1}{3}x^2 dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3.$$

   (e) This can be done either by integration by parts or by substitution. We do it by substitution here. Set $u = x^2 + 1$ so that $du = 2xdx$ and our integral becomes,

$$\int \frac{1}{2}(u - 1)\sqrt{u} du = \frac{1}{2}\int u^{\frac{3}{2}} - \sqrt{u} = \frac{1}{5}u^{\frac{5}{2}} - \frac{1}{3}u^{\frac{3}{2}} + c = \frac{1}{5}(x^2 + 1)^{\frac{5}{2}} - \frac{1}{3}(x^2 + 1)^{\frac{3}{2}} + c.$$

   (f) Once again integrate by parts with $u = x + 1, dv = e^x dx, du = dx, v = e^x$. Doing so yields the solution,

$$\int (x + 1)e^x dx = (x + 1)e^x - \int e^x dx = (x + 1)e^x - e^x + c = xe^x + c.$$

   (g) We first break the integral into two parts across the difference. The first part is $\int \ln(x)dx$, which we integrate by parts. Taking $u = \ln(x), dv = dx, du = \frac{dx}{x}, v = x$, we find,

$$\int \ln(x)dx = x\ln(x) - \int dx = x\ln(x) - x + c.$$

   For the second term we may factor the denominator, and use the substitution $u = x + 1$ to find,

$$\int \frac{dx}{x^2 + 2x + 1} = \int \frac{dx}{(x + 1)^2} = \int \frac{du}{u^2} = \frac{-1}{u} + c = \frac{-1}{x + 1} + c.$$

   Our final answer is therefore $\ln(x) - x + \frac{1}{x+1} + c$.

6.

(a) $\frac{68}{3}$;
(b) $2 - \ln(2)$;
(c) $\ln(5)$;
(d) $9\ln(3) - \frac{26}{9}$;
(e) $\frac{2}{15}\sqrt{2}(125\sqrt{5} - 1)$;
(f) $3e^3 - e$;
(g) $\ln(27) - \frac{9}{4}$.

7. Estimate the following integrals using both Simpson's Rule and the Trapezoid Rule for $n = 2$ and $n = 4$:

(a) For $n = 2$ our interval has cut points $x_0 = 0, x_1 = 1, x_2 = 2$. It follows that the Trapezoid rule approximation is,

$$\frac{2}{4}(f(x_0) + 2f(x_1) + f(x_2)) = \frac{1}{2}(1 + 2e^{-1} + e^{-4})$$

The Simpson's Rule Approximation is,

$$\frac{2}{6}(f(x_0) + 4f(x_1) + f(x_2)) = \frac{1}{3}(1 + 4e^{-1} + e^{-4}).$$

For $n = 4$ our interval has cut points $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2$. It follows that the Trapezoid rule approximation is,

$$\frac{2}{8}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) = \frac{1}{4}(1 + 2e^{-\frac{1}{4}} + 2e^{-1} + 2e^{\frac{-9}{4}} + e^{-4})$$

The Simpson's Rule Approximation is,

$$\frac{2}{12}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) = \frac{1}{6}(1 + 4e^{-\frac{1}{4}} + 2e^{-1} + 4e^{\frac{-9}{4}} + e^{-4}).$$

(b) For $n = 2$ our interval has cut points $x_0 = 1, x_1 = 2, x_2 = 3$. It follows that the Trapezoid rule approximation is,

$$\frac{2}{4}(f(x_0) + 2f(x_1) + f(x_2)) = \frac{1}{2}(\frac{1}{2} + \frac{2}{5} + \frac{1}{10})$$

The Simpson's Rule Approximation is,

$$\frac{2}{6}(f(x_0) + 4f(x_1) + f(x_2)) = \frac{1}{3}(\frac{1}{2} + \frac{4}{5} + \frac{1}{10}).$$

For $n = 4$ our interval has cut points $x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3$. It follows that the Trapezoid rule approximation is,

$$\frac{2}{8}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) = \frac{1}{4}(\frac{1}{2} + \frac{8}{13} + \frac{2}{5} + \frac{8}{25} + \frac{1}{10})$$

The Simpson's Rule Approximation is,

$$\frac{2}{12}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) = \frac{1}{6}(\frac{1}{2} + \frac{16}{13} + \frac{2}{5} + \frac{16}{25} + \frac{1}{10}).$$

(c) For $n = 2$ our interval has cut points $x_0 = -1, x_1 = 0, x_2 = 1$. It follows that the Trapezoid rule approximation is,

$$\frac{2}{4}(f(x_0) + 2f(x_1) + f(x_2)) = \frac{1}{2}(0 + \frac{2}{\ln(2)} + \frac{2}{\ln(3)})$$

The Simpson's Rule Approximation is,

$$\frac{2}{6}(f(x_0) + 4f(x_1) + f(x_2)) = \frac{1}{3}(0 + \frac{4}{\ln(2)} + \frac{2}{\ln(3)}).$$

For $n = 4$ our interval has cut points $x_0 = -1, x_1 = \frac{-1}{2}, x_2 = 0, x_3 = \frac{1}{2}, x_4 = 1$. It follows that the Trapezoid rule approximation is,

$$\frac{2}{8}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) = \frac{1}{4}(0 + \frac{1}{\ln(\frac{9}{4})} + \frac{2}{\ln(2)} + \frac{3}{\ln(\frac{9}{4})} + \frac{2}{\ln(3)})$$

The Simpson's Rule Approximation is,

$$\frac{2}{12}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) = \frac{1}{4}(0 + \frac{2}{\ln(\frac{9}{4})} + \frac{2}{\ln(2)} + \frac{6}{\ln(\frac{9}{4})} + \frac{2}{\ln(3)}).$$