Math 213 Midterm 1 Review

Calculator is not allowed.

1. Find the derivatives of the following functions.
(a) $f(x) = (2x^2 + 3x + 1)^2 + x^2$
(b) $f(x) = \sqrt{4x^2 + 1}$
(c) $f(x) = \frac{2}{\pi^2\sqrt{1-x^2}}$
(d) $f(x) = \frac{e^{-2x}}{2x} - \frac{e}{x}$
(e) $f(x) = x^2(\ln x) - \frac{x^2}{2}$.
(f) Compute $f'(1)$ and $f'(e)$ where $f(x)$ is as in (e).

Solution. (simplification is optional)
(a) $f'(x) = \boxed{2(2x^2 + 3x + 1)(4x + 3) + 2x}$.
(b) $f'(x) = \frac{1}{2}(4x^2 + 1)^{-1/2}(8x) = \boxed{\dfrac{4x}{\sqrt{4x^2 + 1}}}$.
(c) $f'(x) = \frac{2}{\pi^2}(-\frac{1}{2})(1 - x^2)^{-3/2}(-2x) = \boxed{\dfrac{2}{\pi^2}\dfrac{x}{(1 - x^2)^{3/2}}}$.
(d) $f'(x) = \frac{1}{2}\frac{-2e^{-2x}x - e^{-2x}}{x^2} + \frac{e}{x^2} = \boxed{\dfrac{-(2x + 1)e^{-2x} + 2e}{2x^2}}$.
(e) $f'(x) = 2x(\ln x) + x^2\frac{1}{x} - x = \boxed{2x \ln x}$.
(f) $f'(1) = 2 \ln 1 = 0, f'(e) = 2e \ln e = \boxed{2e}$.

2. Find the local maximum and minimum of the function $f(x) = x^3 - 3x + 1$.

Solution. (One can also apply the 1st derivative test.) $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$. Hence the critical points are $x = -1$ and $x = 1$. Since $f''(x) = 6x$, we see that $f''(-1) < 0$ and $f''(1) > 0$. According to the 2nd derivative test, f has local maximum at $x = -1$, and local minimum at $x = 1$. *Final answer:* local maximum at $\boxed{(-1, 3)}$, local minimum at $\boxed{(1, -1)}$.

3. Find the following integrals.
(a) $\int (x + 1)^2 + x + 1dx$
(b) $\int_0^4 \frac{2x}{\sqrt{x^2+9}} - x\sqrt{x^2 + 9}dx$
(c) $\int x^2e^{x^3} + 2dx$
(d) $\int_e^{e^2} \frac{1}{x \ln x}dx$
(e) $\int_0^1 xe^{2x} - xdx$
(f) $\int_1^2 x^2 \ln xdx$.

Solution.
(a) $\int (x + 1)^2 + x + 1dx = \int (x + 1)^2dx + \int xdx + \int 1dx = \boxed{\dfrac{1}{3}(x + 1)^3 + \dfrac{1}{2}x^2 + x + C}$.
(b) $\int_0^4 \frac{2x}{\sqrt{x^2+9}} - x\sqrt{x^2 + 9}dx = \int_0^4 \frac{2x}{\sqrt{x^2+9}}dx - \int_0^4 x\sqrt{x^2 + 9}dx$

$= \int_0^4 \frac{1}{\sqrt{x^2+9}} d(x^2+9) - \frac{1}{2}\int_0^4 \sqrt{x^2+9}\, d(x^2+9) = \int_9^{25} \frac{1}{\sqrt{u}} du - \frac{1}{2}\int_9^{25} \sqrt{u} du$
$= 2\sqrt{u}|_9^{25} - \frac{1}{2}\frac{2}{3}u^{3/2}|_9^{25} = 4 - \frac{1}{3}(125-27) = \boxed{-86/3}$.

(c) $\int x^2 e^{x^3} + 2dx = \int x^2 e^{x^3} dx + \int 2dx = \frac{1}{3}\int e^{x^3} dx^3 + 2x = \frac{1}{3}\int e^u du + 2x = \boxed{\frac{1}{3}e^{x^3} + 2x + C}$

(d) $\int_e^{e^2} \frac{1}{x \ln x} dx = \int_e^{e^2} \frac{1}{\ln x} d \ln x = \int_1^2 \frac{1}{u} du = (\ln u)|_1^2 = \boxed{\ln 2}$.

(e) $\int_0^1 xe^{2x} - xdx = \int_0^1 xe^{2x} dx - \int_0^1 xdx = \frac{1}{2}\int_0^1 xde^{2x} - \frac{1}{2}x^2|_0^1$
$= \frac{1}{2}(xe^{2x}|_0^1 - \int_0^1 e^{2x} dx) - \frac{1}{2} = \frac{1}{2}(e^2 - \frac{1}{2}e^{2x}|_0^1) - \frac{1}{2} = \boxed{\frac{e^2}{4} - \frac{1}{2}}$.

(f) $\int_1^2 x^2 \ln x dx = \frac{1}{3}\int_1^2 \ln x dx^3 = \frac{1}{3}x^3 \ln x|_1^2 - \frac{1}{3}\int_1^2 \frac{1}{x} x^3 dx$
$= \frac{8}{3}\ln 2 - \frac{1}{3}\int_1^2 x^2 dx = \frac{8}{3}\ln 2 - \frac{1}{9}x^3|_1^2 = \boxed{\frac{8}{3}\ln 2 - \frac{7}{9}}$.

3. Use the trapezoidal rule and Simpson's rule to approximate the value of the definite integral for $n = 4$.

$$\int_0^4 x^2 dx.$$

Solution. Trapezoidal rule:

$$\int_0^4 x^2 dx \approx \frac{(4-0)}{2 \cdot 4}\left[0^2 + 2\cdot 1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 + 4^2\right] = \boxed{22}.$$

Simpson's rule:

$$\int_0^4 x^2 dx \approx \frac{(4-0)}{3 \cdot 4}\left[0^2 + 4\cdot 1^2 + 2 \cdot 2^2 + 4 \cdot 3^2 + 4^2\right] = \boxed{\frac{64}{3}}.$$

4. Let $A = \langle 1, 3, 2\rangle$, $B = \langle 2, 5, 4\rangle$, $C = \langle -1, 2, 4\rangle$ be three points in the space.
(a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
(b) Find the distances between A and B and between A and C.
(c) Use inner product to find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
(d) Verify that

$$\|\overrightarrow{AB} + \overrightarrow{AC}\|^2 + \|\overrightarrow{AB} - \overrightarrow{AC}\|^2 = 2\|\overrightarrow{AB}\|^2 + 2\|\overrightarrow{AC}\|^2.$$

Solution.
(a) $\overrightarrow{AB} = \boxed{(1,2,2)}$, $\overrightarrow{AC} = \boxed{(-2,-1,2)}$.
(b) $\|\overrightarrow{AB}\| = \|(1,2,2)\| = \sqrt{1^2+2^2+2^2} = \boxed{3}$,
$\|\overrightarrow{AC}\| = \|(-2,-1,2)\| = \sqrt{(-2)^2+(-1)^2+2^2} = \boxed{3}$.
(c) $\overrightarrow{AB} \cdot \overrightarrow{AC} = (1,2,2)\cdot(-2,-1,2) = 1\cdot(-2) + 2\cdot(-1) + 2\cdot 2 = \boxed{0}$. Hence the angle is $\boxed{90^\circ}$.
(d) $\overrightarrow{AB} + \overrightarrow{AC} = (1,2,2) + (-2,-1,2) = (-1,1,4)$, $\overrightarrow{AB} - \overrightarrow{AC} = (1,2,2) - (-2,-1,2) = (3,3,0)$. The rest follows from direct checking.

5. (a) Find the plane perpendicular to $n = (-1,1,2)$ and going through the point $P = (2,-2,-4)$. (b) Find the intercepts of the plane with the axes.

Solution. (a) The equation of the plane is

$$\boxed{-x+y+2z} = -(2)+(-2)+2(-4) = \boxed{-12}.$$

(b) x-intercept: $\boxed{(12,0,0)}$, y-intercept: $\boxed{(0,-12,0)}$, z-intercept: $\boxed{(0,0,-6)}$.

6. Give examples of equations describing the following types of surfaces: (a) ellipsoid (b) elliptic cone (c) elliptic paraboloid (d) hyperbolic paraboloid (e) hyperboloid of one sheet (f) hyperboloid of two sheets.

Solution. (a) $x^2+4y^2+9z^2=1$ (b) $x^2=4y^2+9z^2$ (c) $x=4y^2+9z^2$ (d) $x=4y^2-9z^2$ (e) $x^2+4y^2-9z^2=1$ (f) $x^2-4y^2-9z^2=1$.

7. Let $f(x,y)=e^{-x^2-y^2}$ and $g(x,y)=\frac{xy}{x^2+y^2}$.
(a) Find f_{xy} and f_{xx}.
(b) Find $\partial_y g(1,1)$.

Solution. (a)

$$f_x(x,y) = -2xe^{-x^2-y^2},\, f_{xy}(x,y) = \boxed{4xye^{-x^2-y^2}}.$$

$$f_{xx}(x,y) = -2e^{-x^2-y^2}+4x^2e^{-x^2-y^2} = \boxed{(4x^2-2)e^{-x^2-y^2}}.$$

(b)

$$\partial_y g(x,y) = \frac{x(x^2+y^2)-xy(2y)}{(x^2+y^2)^2} = \frac{x^3-xy^2}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

Hence

$$\partial_y g(1,1) = \boxed{0}.$$