Math 213 Final Exam Review - Part 2 Solutions

1. Solving

$$z_x = 2x + y - 2 = 0$$
$$z_y = 2y + x + 2 = 0$$

gives $\boxed{x = 2}$ and $\boxed{y = -2}$. Plug in to z, we get

$$\boxed{z_{\min} = -4}.$$

2. Since $\ln(u)$ strictly increasing in u, we may just look at the function

$$z = 1 + x^2 + y^2.$$

Solving

$$z_x = 2x = 0$$
$$z_y = 2y = 0$$

gives $x = 0$ and $y = 0$. This shows that $\boxed{(x, y) = (0, 0)}$ is the unique critical point. To determine the type, we find

$$z_{xx} = 2$$
$$z_{yy} = 2$$
$$z_{xy} = 0$$

thus the determinant $d = z_{xx}z_{yy} - (z_{xy})^2 > 0$. Since $z_{xx} = 2 > 0$, we conclude that at $(0, 0)$ the function attains a $\boxed{\text{local minimum}}$, according to the second derivative test.

3. Let

$$L = (yz + xz + xy) - \lambda(xyz - 1000)$$

and set the derivatives equal to zero 0, we get

$$L_x = y + z - \lambda yz = 0$$
$$L_y = x + z - \lambda xz = 0$$
$$L_z = x + y - \lambda xy = 0$$
$$L_\lambda = -xyz + 1000 = 0$$

Solve the first equation for y, we get

$$y = \frac{-z}{1 - \lambda z}.$$

Solve the second equation for x, we get

$$x = \frac{-z}{1 - \lambda z}.$$

This shows that $x = y$. Similarly, one can show that $y = z$ from the second and third equations. Now plug in $x = y = z$ to the last equation, we get

$$-x^3 + 1000 = 0$$

and hence $x = 10$. We can now conclude that at $\boxed{(x, y, z) = (10, 10, 10)}$, $S = yz + xz + xy$ attains its minimum $\boxed{S_{\text{min}} = 300}$.

4. Suppose that the equation of the line is

$$f(x) = ax + b.$$

Then the sum of squared errors is given by

$$\begin{aligned} S &= (f(-1) - 1)^2 + (f(0) - 0)^2 + (f(2) - 0)^2 \\ &= (-a + b - 1)^2 + b^2 + (2a + b)^2. \end{aligned}$$

Set the derivatives equal to zero 0, we get

$$S_a = -2(-a + b - 1) + 0 + 4(2a + b) = 0$$

$$S_b = 2(-a + b - 1) + 2b + 2(2a + b) = 0.$$

In other words,

$$2(5a + b + 1) = 0$$

$$2(a + 3b - 1) = 0.$$

Solving these equations we get

$$a = -\frac{2}{7}, \ b = \frac{3}{7}.$$

So the line of best fit is

$$\boxed{f(x) = -\frac{2}{7}x + \frac{3}{7}}.$$

5. Rewrite the equation as

$$x\frac{dy}{dx} = \frac{y^2}{x^2},$$

and further as

$$\frac{dy}{y^2} = \frac{dx}{x^3}.$$

Integrate each side, we get

$$\int y^{-2} dy = \int x^{-3} dx$$

that is, by the power rule,

$$-\frac{1}{y} = -\frac{1}{2x^2} + C.$$

Solving for y, we get

$$\boxed{y = \frac{1}{\frac{1}{2x^2} + C}}.$$

6. Rewrite the equation as

$$\frac{dy}{dx} = x(y+1),$$

and further as

$$\frac{dy}{y+1} = xdx.$$

Integrate each side, we get

$$\ln(y+1) = \frac{x^2}{2} + C.$$

Take the exponential of each side, we get

$$y + 1 = e^{\frac{x^2}{2} + C} = Ce^{\frac{x^2}{2}}.$$

So

$$\boxed{y = Ce^{\frac{x^2}{2}} - 1}.$$

7. To use the method integrating factor, we let

$$P(x) = \frac{1}{x}, \;\; Q(x) = e^{-x^2}.$$

Then

$$u(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

and thus

$$\begin{aligned} y &= \frac{1}{u(x)}\Big(\int u(x)Q(x)dx\Big) \\ &= \frac{1}{x}\Big(\int xe^{-x^2}dx\Big) \\ &= \frac{1}{x}\Big(\frac{1}{-2}\int e^{-x^2}d(-x^2)\Big) \\ &= \frac{1}{x}\Big(-\frac{1}{2}\int e^{u}du\Big) \\ &= \frac{1}{x}\Big(-\frac{1}{2}e^{u} + C\Big) \\ &= \frac{1}{x}\Big(-\frac{1}{2}e^{-x^2} + C\Big). \end{aligned}$$

On the other hand, since $y = 0$ when $x = 1$, we see that

$$0 = -\frac{1}{2}e^{-1} + C$$

i.e.

$$C = \frac{1}{2}e^{-1}.$$

So

$$y = \frac{1}{x}\Big(-\frac{1}{2}e^{-x^2} + \frac{1}{2}e^{-1}\Big) = \boxed{\frac{1}{2x}\big(- e^{-x^2} + e^{-1}\big)}.$$

8. a) The sample space is

$$\Big\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\Big\}.$$

b) The event that there are exactly two heads consists of

$$\Big\{HHT, HTH, THH\Big\}.$$

c) $\boxed{\frac{3}{8}}$.

d) We have the following table for the probability distribution of x.

x	0	1	2	3
$P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Hence the expected value of x is

$$\mu = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \boxed{\frac{3}{2}}.$$

e) The variance of x is given by

$$(0 - \frac{3}{2})^2 \cdot \frac{1}{8} + (1 - \frac{3}{2})^2 \cdot \frac{3}{8} + (2 - \frac{3}{2})^2 \cdot \frac{3}{8} + (3 - \frac{3}{2})^2 \cdot \frac{1}{8} = \boxed{\frac{3}{4}}.$$

9. a)

$$P(x \leq 1) = \int_0^1 \frac{3}{4}x(2-x)dx = \frac{3}{4}\int_0^1 (2x - x^2)dx = \frac{3}{4}\big[x^2 - \frac{x^3}{3}\big]_0^1 = \boxed{\frac{1}{2}}.$$

b)

$$\mu = \int xf(x)dx = \int_0^2 x \cdot \frac{3}{4}x(2-x)dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)dx = \boxed{1}.$$

c)

$$V = \Big(\int x^2 f(x)dx\Big) - \mu^2 = \Big(\int_0^2 x^2 \cdot \frac{3}{4}x(2-x)dx\Big) - 1 = \boxed{\frac{1}{5}}.$$

d) $\sigma = \sqrt{V} = \boxed{\frac{1}{\sqrt{5}}}$.

10. a) We should have

$$\int_0^1 ke^{-x}dx = 1.$$

But the left hand side is equal to

$$k\int_0^1 e^{-x}dx = k\Big[-e^{-x}\Big]_0^1 = k(-e^{-1}+1).$$

So

$$\boxed{k = \frac{1}{1-e^{-1}} \approx 1.582}.$$

b)

$$\begin{aligned}\mu &= \int xf(x)dx = \int_0^1 x(ke^{-x})dx = k\int_0^1 xe^{-x}dx\\ &= k\int_0^1 xd(-e^{-x}) = k\Big(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}dx\Big)\\ &= k\Big(-e^{-1} + [-e^{-x}]_0^1\Big) = k(-2e^{-1}+1)\\ &= \boxed{\frac{1-2e^{-1}}{1-e^{-1}} \approx 0.418}.\end{aligned}$$

c)

$$\begin{aligned}V &= \int x^2 f(x)dx - \mu^2 = \int_0^1 x^2(ke^{-x})dx - \mu^2\\ &= k\int_0^1 x^2e^{-x}dx - \mu^2 \text{ (integrate by parts twice...)}\\ &= k(2-5e^{-1}) - \mu^2\\ &\approx \boxed{0.079}.\end{aligned}$$

11. a) hyperbolic paraboloid
b) paraboloid
c) none of the above (elliptic cone)
d) ellipsoid
e) hyperboloid of one sheet
f) hyperboloid of two sheets

12. Using step size 1, we have

$$\begin{aligned}y(0) &= 1\\ y(1) &\approx y(0) + 1\cdot y'(0)\\ &= 1 + (y^2(0)+0)\\ &\approx 1 + (1^2+0)\\ &= 2\\ y(2) &\approx y(1) + 1\cdot y'(1)\end{aligned}$$

$$\begin{aligned} &\approx 2 + (y^2(1) + 1) \\ &\approx 2 + (2^2 + 1) \\ &= \boxed{7}. \end{aligned}$$

Using step size 1/2, we have

$$\begin{aligned} y(0) &= 1 \\ y(\frac{1}{2}) &\approx y(0) + \frac{1}{2}y'(0) \\ &= 1 + \frac{1}{2}(y^2(0) + 0) \\ &\approx 1 + \frac{1}{2}(1^2 + 0) \\ &= \frac{3}{2} \\ y(1) &\approx y(\frac{1}{2}) + \frac{1}{2}y'(\frac{1}{2}) \\ &\approx \frac{3}{2} + \frac{1}{2}(y^2(\frac{1}{2}) + \frac{1}{2}) \\ &\approx \frac{3}{2} + \frac{1}{2}((\frac{3}{2})^2 + \frac{1}{2}) \\ &= \frac{23}{8}. \\ y(\frac{3}{2}) &\approx y(1) + \frac{1}{2}y'(1) \\ &\approx \frac{23}{8} + \frac{1}{2}(y^2(1) + 1) \\ &\approx \frac{23}{8} + \frac{1}{2}((\frac{23}{8})^2 + 1) \\ &= \frac{961}{128}. \\ y(2) &\approx y(\frac{3}{2}) + \frac{1}{2}y'(\frac{3}{2}) \\ &\approx \frac{961}{128} + \frac{1}{2}(y^2(\frac{3}{2}) + \frac{3}{2}) \\ &\approx \frac{961}{128} + \frac{1}{2}((\frac{961}{128})^2 + \frac{3}{2}) \\ &\boxed{\approx 36.44}. \end{aligned}$$

13. Let $A(t)$ be the amount of radioactive carbon present after t years. Since the rate of decrease is proportional to $A(t)$, we have

$$\frac{dA}{dt} = -kA$$

for some positive constant k. Solving this equation by separating the variables we get

$$A(t) = Ce^{-kt}$$

for some positive constant C.

Now since the half-life of radioactive carbon is 5715 year, we have

$$A(5715) = \frac{1}{2}A(0)$$

i.e.

$$Ce^{-5715k} = \frac{1}{2}C.$$

From this we solve

$$k = \frac{\ln 2}{5715}.$$

Thus the percentage of the present amount that will remain after 1000 years is

$$\frac{A(1000)}{A(0)} = \frac{Ce^{-1000k}}{C} = e^{-1000\frac{\ln 2}{5715}} \approx 0.88 = \boxed{88\%}.$$