Math 213 Discussion Worksheet – Week 8

1. (a) Find ∇F where $F(x,y) = x^2 + y^2 - 2xy$. (b) Find the rate of change of F along the direction of the vector $\vec{v} = (1,1)$.

(a) $\nabla F = (F_x, F_y) = \boxed{(2x-2y,\ 2y-2x)}$

(b) $\nabla_{\hat{v}} F = \nabla F \cdot \frac{\vec{v}}{\|\vec{v}\|} = (2x-2y, 2y-2x)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \boxed{0}$

2. Consider a box of dimensions x, y, z. (a) Maximize the volume of the box $V = xyz$ under the constraint $x + y + z = 3$. (b) Maximize V under the constraint $yz + xz + xy = 3$.

(a) $F(x,y,z,\lambda) = xyz - \lambda(x+y+z-3)$

$$\begin{cases} F_x = yz - \lambda = 0 & (1) \\ F_y = xz - \lambda = 0 & (2) \\ F_z = xy - \lambda = 0 & (3) \end{cases}$$
$F_\lambda = -x-y-z+3 = 0 \quad (4)$

$(1),(2),(3) \Rightarrow yz = xz = xy (= \lambda)$

$\xRightarrow{\text{(multiply } \frac{1}{xyz})} \frac{1}{x} = \frac{1}{y} = \frac{1}{z}$

$\Rightarrow x = y = z$

$(4) \Rightarrow -3x + 3 = 0 \Rightarrow x = 1$

$\Rightarrow (x,y,z) = \boxed{(1,1,1)}$

$\Rightarrow V_{max} = \boxed{1}$

(b) $F(x,y,z,\lambda) = xyz - \lambda(yz + xz + xy - 3)$

$$\begin{cases} F_x = yz - \lambda(z+y) = 0 & (1') \\ F_y = xz - \lambda(x+z) = 0 & (2') \\ F_z = xy - \lambda(x+y) = 0 & (3') \\ F_\lambda = -yz - xz - xy + 3 = 0 & (4') \end{cases}$$

$(1') \Rightarrow (y-\lambda)(z-\lambda) = \lambda^2$

$(2') \Rightarrow (x-\lambda)(z-\lambda) = \lambda^2$

$(3') \Rightarrow (x-\lambda)(y-\lambda) = \lambda^2$

$(1'),(2'),(3') \Rightarrow x-\lambda = y-\lambda = z-\lambda$ (as before)

$\Rightarrow x = y = z$

$(4') \Rightarrow -3x^2 + 3 = 0 \Rightarrow x = 1 \Rightarrow y = 1, z = 1$

$\Rightarrow (x,y,z) = \boxed{(1,1,1)}$, $V_{max} = \boxed{1}$

3. Consider a box of dimensions x, y, z. (a) Maximize the area of the box $A = 2yz + 2xz + 2xy$ under the constraint $x + y + z = 3$. (b) Minimize the area under the constraint $xyz = 1$.

(a) $F(x,y,z,\lambda) = 2yz + 2xz + 2xy - \lambda(x+y+z-3)$

$$\begin{cases} F_x = 2z + 2y - \lambda = 0 & (1) \\ F_y = 2z + 2x - \lambda = 0 & (2) \\ F_z = 2y + 2x - \lambda = 0 & (3) \\ F_\lambda = -x-y-z+3 = 0 & (4) \end{cases}$$

$(1)+(2)+(3) \Rightarrow 4(x+y+z) = 3\lambda$

$\Rightarrow 2x + 2y + 2z = \frac{3}{2}\lambda \quad (*)$

$(*) - (1) \Rightarrow 2x = \frac{1}{2}\lambda \Rightarrow x = \frac{1}{4}\lambda$

$(*) - (2) \Rightarrow y = \frac{1}{4}\lambda$

$(*) - (3) \Rightarrow z = \frac{1}{4}\lambda$

$\Rightarrow x = y = z$

$(4) \Rightarrow -3x + 3 = 0 \Rightarrow x = 1$

$\Rightarrow (x,y,z) = \boxed{(1,1,1)}$

$S_{max} = \boxed{6}$

(b) $F(x,y,z,\lambda) = 2yz + 2xz + 2xy - \lambda(xyz - 1)$

$$\begin{cases} F_x = 2z + 2y - \lambda yz = 0 & (1') \\ F_y = 2z + 2x - \lambda xz = 0 & (2') \\ F_z = 2y + 2x - \lambda xy = 0 & (3') \\ F_\lambda = -xyz + 1 = 0 & (4') \end{cases}$$

$$\begin{cases} (1') \Rightarrow yz - 2\lambda^{-1}z - 2\lambda^{-1}z = 0 \\ (2') \Rightarrow xz - 2\lambda^{-1}z - 2\lambda^{-1}x = 0 \\ (3') \Rightarrow xy - 2\lambda^{-1}y - 2\lambda^{-1}x = 0 \end{cases}$$

$$\begin{cases} (y - 2\lambda^{-1})(z - 2\lambda^{-1}) = 4\lambda^{-2} \\ (x - 2\lambda^{-1})(z - 2\lambda^{-1}) = 4\lambda^{-2} \\ (x - 2\lambda^{-1})(y - 2\lambda^{-1}) = 4\lambda^{-2} \end{cases}$$

$\Rightarrow x = y = z$ (as in 2.(b))

$(4) \Rightarrow x = y = z = 1$, $(x,y,z) = \boxed{(1,1,1)}$

$S_{min} = \boxed{6}$