Math 213 Discussion Worksheet – Week 9

1. Let $F(x,y) = x^2 - y^2 - 2$. Find the derivative of F in the direction of the vector $\overrightarrow{v} = (1,3)$ at the point (3,2).

Solution. The normalized vector of $\overrightarrow{v}$ is

$$\overrightarrow{u} = \frac{\overrightarrow{v}}{\|\overrightarrow{v}\|} = \frac{1}{\sqrt{10}}(1,3) = (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}).$$

The gradient of F is

$$\nabla F = (F_x, F_y) = (2x, -2y).$$

Evaluating at the given point $(x,y) = (3,2)$ we get

$$\nabla F\Big|_{(3,2)} = (6, -4).$$

Hence the derivative of F in $\overrightarrow{v}$ direction at $(x,y) = (3,2)$ is

$$\nabla F\Big|_{(3,2)} \cdot \overrightarrow{u} = (6,-4) \cdot (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}) = \boxed{\frac{-6}{\sqrt{10}}}.$$

2. Consider the following set of data points $\{(0,0),(1,1),(2,4),(3,9)\}$. (a) Find the the line of best fit, and compute the corresponding sum of squared errors. (b)* Find the quadratic of best fit, and compute the corresponding sum of squared errors.

Solution. (a) Suppose the linear model is $f(x) = ax + b$, then the sum of squared errors is

$$S = b^2 + (a+b-1)^2 + (2a+b-4)^2 + (3a+b-9)^2.$$

If S is minimized with coefficients a and b, then

$$\frac{\partial S}{\partial a} = 2(a+b-1) + 4(2a+b-4) + 6(3a+b-9) = 28a + 12b - 72 = 0$$
$$\frac{\partial S}{\partial b} = 2b + 2(a+b-1) + 2(2a+b-4) + 2(3a+b-9) = 12a + 8b - 28 = 0.$$

i.e.

$$7a + 3b = 18$$
$$3a + 2b = 7.$$

Solving this system gives $\boxed{a = 3}$, $\boxed{b = -1}$. The corresponding sum of squared errors is

$$S(3,-1) = \boxed{4}.$$

(b)* Suppose the linear model is $f(x) = ax^2 + bx + c$, then the sum of squared errors is

$$S = c^2 + (a + b + c - 1)^2 + (4a + 2b + c - 4)^2 + (9a + 3b + c - 9)^2.$$

If S is minimized with coefficients a, b and c, then

$$\begin{aligned}
\frac{\partial S}{\partial a} &= 2(a + b + c - 1) + 8(4a + 2b + c - 4) + 18(9a + 3b + c - 9) = 0 \\
\frac{\partial S}{\partial b} &= 2(a + b + c - 1) + 4(4a + 2b + c - 4) + 6(9a + 3b + c - 9) = 0 \\
\frac{\partial S}{\partial c} &= 2c + 2(a + b + c - 1) + 2(4a + 2b + c - 4) + 2(9a + 3b + c - 9) = 0.
\end{aligned}$$

i.e.

$$\begin{aligned}
49a + 18b + 7c &= 49 \\
18a + 7b + 3c &= 18 \\
7a + 3b + 2c &= 7.
\end{aligned}$$

Solving this system gives $\boxed{a = 1}$, $\boxed{b = 0}$, $\boxed{c = 0}$. The corresponding sum of squared errors is

$$S(1, 0, 0) = \boxed{0}.$$