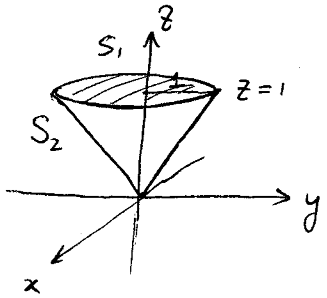
Name: ______________________________ Math 234 Quiz 11

Section: 328 ☐ 329 ☐ Dec 11, 2014

Solve one of the following two problems. Problem 2 on the back.

1. Let S be the boundary of the region

$$R = \{(x,y,z) : \underbrace{\sqrt{x^2+y^2}}_{=r} \le z \le 1\}.$$



Compute the flux integral

$$\iint_S \vec{v}\cdot\vec{N}\,dA$$

where $\vec{v} = x^2\vec{\imath} + y^2\vec{\jmath} + z\vec{k}$ and $\vec{N}$ is the outward normal.

Solution 1 (Divergence thm):

$$\iint_S \vec{v}\cdot\vec{N}dA = \iiint_R \operatorname{div}\vec{v}\,dV = \iiint_R (x^2)_x + (y^2)_y + (z)_z\,dV = \iiint_R 2x+2y+1\,dV$$

Cylindrical Coord.

$$= \int_0^{2\pi}\int_0^1\int_{r}^{1} (2r\cos\theta + 2r\sin\theta + 1)\,dz\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r(1-r)\cos\theta\; r\,dr\,d\theta + \int_0^{2\pi}\int_0^1 2r(1-r)\sin\theta\; r\,dr\,d\theta + \int_0^{2\pi}\int_0^1 (1-r)\,r\,dr\,d\theta$$

Switch order

$$= \int_0^1\int_0^{2\pi} 2r^2(1-r)\cos\theta\,d\theta\,dr + \int_0^1\int_0^{2\pi} 2r^2(1-r)\sin\theta\,d\theta\,dr + \int_0^{2\pi}\int_0^1 (r-r^2)\,dr\,d\theta$$

$\int_0^{2\pi}\cos\theta\,d\theta = 0$

$$= 0 + 0 + \int_0^{2\pi}\left(\tfrac{1}{2}-\tfrac{1}{3}\right)d\theta$$

$$= \boxed{\frac{\pi}{3}}$$

Solution 2 (direct computation)

$$\iint_S \vec{v}\cdot\vec{N}\,dA = \iint_{S_1}\vec{v}\cdot\vec{N}dA + \iint_{S_2}\vec{v}\cdot\vec{N}dA \quad \text{where } S_1 = \text{top},\ S_2 = \text{side}.$$

on S_1: $\vec{N} = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $dA = dxdy$, $\overset{z=1}{\wedge}$ so $\iint_{S_1}\vec{v}\cdot\vec{N}dA = \iint_{S_1}\begin{pmatrix}x^2\\y^2\\z\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}dxdy = \iint_{S_1} 1\,dxdy = \pi$ (area of S_1)

S_2: given by the graph $z = \sqrt{x^2+y^2}$, so

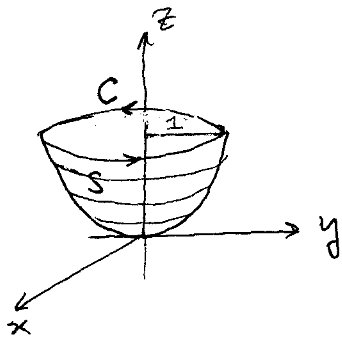
$$\vec{N}dA = \begin{pmatrix}-z_x\\-z_y\\1\end{pmatrix}dxdy = \begin{pmatrix}-\frac{x}{\sqrt{x^2+y^2}}\\-\frac{y}{\sqrt{x^2+y^2}}\\1\end{pmatrix}dxdy$$

$$\iint_{S_2}\vec{v}\cdot\vec{N}dA = \iint\left(\frac{-x^3}{\sqrt{x^2+y^2}} + \frac{-y^3}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\right)dxdy$$

$$= \int_0^{2\pi}\int_0^1 \cdots\, r\,dr\,d\theta \text{ (use polar coordinates).}$$

2. Consider the surface patch

$$S = \{(x, y, z) : z = x^2 + y^2,\ x^2 + y^2 \le 1\}.$$



Evaluate the flux integral

$$\iint_S (\mathbf{curl}\,\vec{F}) \cdot \vec{N}\, dA$$

where $\vec{F} = -y\,\vec{\imath} + x\,\vec{\jmath} + z^2\,\vec{k}$ and $\vec{N}$ is the upward normal.

Solution 1 (Stokes thm)

Let $C = \{(x,y,z) : x^2+y^2=1,\ z=1\}$, then S has boundary C.

By Stokes thm,

$$\iint_S (\operatorname{curl}\vec{F})\cdot\vec{N}\,dA = \oint_C \vec{F}\cdot d\vec{x}$$

We can parametrize C by $\vec{x}(t) = (\cos t, \sin t, 1)$, $0 \le t \le 2\pi$. Then

$$\oint_C \vec{F}\cdot d\vec{x} = \int_0^{2\pi} \begin{pmatrix} -y \\ x \\ z^2 \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \int_0^{2\pi} -y\,dx + x\,dy + z^2\,dz$$

$$= \int_0^{2\pi} -\sin t\,(-\sin t)\,dt + \cos t \sin t\,dt + \cancel{1\,d1}^{\,0}$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$$

Solution 2 (direct computation)

S is given by the graph $z = x^2+y^2$, so

$$\vec{N}\,dA = \begin{pmatrix} -z_x \\ -z_y \\ 1 \end{pmatrix} dx\,dy = \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix} dx\,dy$$

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} -y \\ x \\ z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

So $$\iint_S (\operatorname{curl}\vec{F})\cdot\vec{N}\,dA = \iint_D \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix} dx\,dy \qquad \left(D = \{(x,y),\ x^2+y^2 \le 1\}\right)$$

$$= \iint_D 2\,dx\,dy$$

$$= 2\ \text{Area}(D)$$

$$= \boxed{2\pi}$$