Name: ____________________

Section: 328 ☐ 329 ☐

Math 234 Quiz 2

Sep 16, 2014

1. (10 pts) Consider the curve given by

$$\vec{x}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}.$$

(a) Compute the velocity, acceleration and jerk (third derivative) vectors. (b) Find the volume of the parallelepiped spanned by these three vectors.

(a) $\vec{x}' = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$

$\vec{x}'' = \begin{pmatrix} 0 \\ 2 \\ 6t \end{pmatrix}$

$\vec{x}''' = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$

(b) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 0 & 0 \\ 2t & 2 & 0 \\ 3t^2 & 6t & 6 \end{vmatrix}$

$= 1\begin{vmatrix} 2 & 0 \\ 6t & 6 \end{vmatrix} - 0\cancel{\begin{vmatrix} 2t & 0 \\ 3t^2 & 6 \end{vmatrix}}^{0} + 0\cancel{\begin{vmatrix} 2t & 2 \\ 3t^2 & 6t \end{vmatrix}}^{0}$

$= \boxed{12}$

2. (10 pts) Consider the curve given by

$$\vec{x}(\theta) = \begin{pmatrix} \cos\theta + \sin\theta \\ \cos\theta - \sin\theta \\ \theta \end{pmatrix}.$$

Compute the length of the segment with $0 \le \theta \le 2\pi$.

$\vec{x}'(\theta) = \begin{pmatrix} -\sin\theta + \cos\theta \\ -\sin\theta - \cos\theta \\ 1 \end{pmatrix}$

$\|x'(\theta)\| = \sqrt{(-\sin\theta + \cos\theta)^2 + (-\sin\theta - \cos\theta)^2 + 1^2}$

$= \sqrt{(\sin^2\theta - \cancel{2\sin\theta\cos\theta} + \cos^2\theta) + (\sin^2\theta + \cancel{2\sin\theta\cos\theta} + \cos^2\theta) + 1}$

$= \sqrt{2\sin^2\theta + 2\cos^2\theta + 1} = \sqrt{2+1} = \sqrt{3}$

length $= \int_0^{2\pi} \|\vec{x}'(\theta)\| d\theta = \int_0^{2\pi} \sqrt{3}\, d\theta = \boxed{\sqrt{3}\cdot 2\pi}$

Bonus. (5 pts) Consider the curve given in Problem 1. (a) Compute the curvature at $t = 0$. (b) Find the limit of the curvature as $t \to \infty$.

(a) $\kappa(t) = \frac{1}{\|\vec{x}'\|^3}\|\vec{x}'\times\vec{x}''\| = \frac{1}{(\sqrt{1+4t^2+9t^4})^3}\left\| \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 6t \end{pmatrix} \right\|$

$= \frac{1}{(1+4t^2+9t^4)^{3/2}} \left\| \begin{pmatrix} 6t^2 \\ -6t \\ 2 \end{pmatrix} \right\| = \frac{2(9t^4+9t^2+1)^{1/2}}{(1+4t^2+9t^4)^{3/2}} \overset{t=0}{=} \boxed{2}$

(b) $\lim_{t\to\infty} \frac{2(9t^4+9t^2+1)^{1/2}}{(9t^4+4t^2+1)^{3/2}} = \boxed{0}$