Name: ______________________ Math 234 Quiz 9

Section: 328 ☐ 329 ☐ Nov 25, 2014

1. (10 pts) Compute $\int_C y^2 ds$ where C is the curve $y = e^x, 0 \le x \le 1$.
2. (10 pts) Compute $\oint_C x^2dx + y^2dy$ where C is the unit circle oriented counter-clockwise.
Bonus. (5 pts) Compute $\oint_C y^2dx + x^2dy$ where C is as in Problem 2.

1. C: $x(t) = t$, $y(t) = e^t$, $0 \le t \le 1$.

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1^2 + (e^t)^2} = \sqrt{1+e^{2t}}$$

$$\int_C y^2 ds = \int_0^1 e^{2t}\sqrt{1+e^{2t}}\, dt$$

$$\underset{du = 2e^{2t}dt}{\overset{u = 1+e^{2t}}{=}} \int_2^{1+e^2} \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\frac{u^{3/2}}{3/2}\Big|_2^{1+e^2} = \boxed{\frac{1}{3}\left((1+e^2)^{3/2} - 2^{3/2}\right)}$$

2. C: $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le 2\pi$.

$$\oint_C x^2dx + y^2dy = \int_0^{2\pi} (\cos t)^2 d(\cos t) + (\sin t)^2 d(\sin t)$$

$$= -\int_0^{2\pi} (\cos t)^2 \sin t\, dt + \int_0^{2\pi} (\sin t)^2 \cos t\, dt$$

$$\underset{du = \sin t\, dt}{\overset{u = \cos t}{=}} \frac{(\cos t)^3}{3}\Big|_0^{2\pi} + \frac{(\sin t)^3}{3}\Big|_0^{2\pi} = \boxed{0}$$

3.

$$\oint_C y^2dx + x^2dy = \int_0^{2\pi} -(\sin t)^3 + (\cos t)^3 dt$$

$$= \int_0^{2\pi} -(\sin t)^2 \sin t + (\cos t)^2 \cos t\ dt$$

$$= \int_0^{2\pi} (\cos^2 t - 1)\sin t + (1 - \sin^2 t)\cos t\, dt$$

$$= \int_0^{2\pi} \cos^2 t \sin t - \sin t + \cos t - \sin^2 t \cos t\, dt$$

$$= \left[-\frac{(\cos t)^3}{3} + \cos t + \sin t - \frac{(\sin t)^3}{3}\right]_0^{2\pi}$$

$$= \boxed{0}$$

Name: ______________________ Math 234 Quiz 9

Section: 328 ☐ 329 ☐ Nov 20, 2014

1. (10 pts) Compute $\int_C y ds$ where C is the curve $y = x^3, 0 \le x \le 1$.
2. Suppose a wire C is the quarter of the unit circle in the first quadrant, and is of constant density 1. (1) (10 pts) Find the center of mass of the wire. (2) (5 pts, bonus) what if the density is y^2?

1. C is given by $\vec{x}(t) = \begin{pmatrix} t \\ t^3 \end{pmatrix}$, $0 \le t \le 1$. So,

$$\int_C y\,ds = \int_0^1 t^3 \|\vec{x}'(t)\| dt = \int_0^1 t^3 \sqrt{1+(3t^2)^2}\,dt = \int_0^1 t^3 \sqrt{1+9t^4}\,dt$$

$$\underset{du = 36t^3 dt}{\overset{u = 1+9t^4}{=}} \frac{1}{36}\int_1^{10} \sqrt{u}\,du = \frac{1}{36}\frac{2}{3} u^{3/2}\Big|_1^{10} = \boxed{\frac{1}{54}\left(10^{3/2} - 1\right)}$$

2.(1) By symmetry, $\bar{x} = \bar{y}$. To find $\bar{x}$, we need to compute

$\bar{x} = \dfrac{\int_C x\,ds}{\int_C 1\,ds}$, where $\int_C 1 ds = \text{length}(C) = \frac{\pi}{2}$. Use parametization $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$,

$$\int_C x\,ds = \int_0^{\pi/2} \cos t \,\|\vec{x}'(t)\|\,dt = \int_0^{\pi/2} \cos t\,dt = \sin t\Big|_0^{\pi/2} = \boxed{1}.$$

So $\bar{x} = \frac{1}{\frac{\pi}{2}} = \boxed{\frac{2}{\pi}} \approx 0.6 = \bar{y}$. The center of mass is $\boxed{\left(\frac{2}{\pi}, \frac{2}{\pi}\right)}$.

(2) $\bar{x} = \dfrac{\int_C x y^2 ds}{\int_C y^2 ds}$, $\bar{y} = \dfrac{\int_C y\, y^2 ds}{\int_C y^2 ds}$.

$$\int_C y^2 ds = \int_0^{\pi/2} (\sin t)^2 dt = \int_0^{\frac{\pi}{2}} \frac{1-\cos(2t)}{2} dt = \frac{\pi}{4} - \left[\frac{\sin(2t)}{4}\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$\int_C x y^2 ds = \int_0^{\pi/2} (\cos t)(\sin t)^2 dt = \frac{(\sin t)^3}{3}\Big|_0^{\pi/2} = 1/3$$

$$\int_C y^3 ds = \int_0^{\pi/2} (\sin t)^3 dt = \int_0^{\pi/2} (1-\cos^2 t)\sin t\,dt \overset{u=\cos t}{=} \int_0^1 (1-u^2)\,du = \frac{2}{3}$$

So, $\bar{x} = \dfrac{\frac{1}{3}}{\frac{\pi}{4}} = \boxed{\frac{4}{3\pi}}$

$\bar{y} = \dfrac{\frac{2}{3}}{\frac{\pi}{4}} = \boxed{\frac{8}{3\pi}}$