Math 234 Review - Answers and Solutions

Chapter 2: Parametric curves and vector functions

1. (1)

$$\overrightarrow{\mathbf{x}}'(t) = \begin{pmatrix} -\omega R\sin(\omega t) \\ \omega R\cos(\omega t) \\ a \end{pmatrix}$$

$$\overrightarrow{\mathbf{x}}''(t) = \begin{pmatrix} -\omega^2 R\cos(\omega t) \\ -\omega^2 R\sin(\omega t) \\ 0 \end{pmatrix}$$

$$\overrightarrow{\mathbf{x}}'''(t) = \begin{pmatrix} \omega^3 R\sin(\omega t) \\ -\omega^3 R\cos(\omega t) \\ 0 \end{pmatrix}$$

(2) $\|\overrightarrow{\mathbf{x}}'(t)\| = \boxed{\sqrt{\omega^2R^2 + a^2}}$

(3) $\int_0^\pi \|\overrightarrow{\mathbf{x}}'(t)\| dt = \boxed{\pi\sqrt{\omega^2R^2 + a^2}}$

(4)* Since $\overrightarrow{\mathbf{x}}'(t) \cdot \overrightarrow{\mathbf{x}}''(t) = 0$, the angle between them is $\frac{\pi}{2}$ or $90°$.

(5)* $\overrightarrow{\mathbf{x}}'(t) \cdot (\overrightarrow{\mathbf{x}}''(t) \times \overrightarrow{\mathbf{x}}'''(t)) = a\omega^5R^2$

(6)

$$\boxed{\overrightarrow{\mathbf{T}}(t) = \frac{\overrightarrow{\mathbf{x}}'(t)}{\|\overrightarrow{\mathbf{x}}'(t)\|}} = \boxed{\frac{1}{\sqrt{\omega^2R^2 + a^2}} \begin{pmatrix} -\omega R\sin(\omega t) \\ \omega R\cos(\omega t) \\ a \end{pmatrix}}$$

(7)

$$\boxed{\overrightarrow{\kappa}(t) = \frac{1}{\|\overrightarrow{\mathbf{x}}'(t)\|} \frac{d}{dt}\overrightarrow{\mathbf{T}}(t)} = \boxed{\frac{1}{\omega^2R^2 + a^2} \begin{pmatrix} -\omega^2 R\cos(\omega t) \\ -\omega^2 R\sin(\omega t) \\ 0 \end{pmatrix}}$$

$$\boxed{\kappa(t) = \|\overrightarrow{\kappa}(t)\|} = \boxed{\frac{\omega^2 R}{\omega^2R^2 + a^2}}$$

2.*

$$\overrightarrow{\mathbf{x}}'(t) = \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}$$

$$\|\overrightarrow{\mathbf{x}}'(t)\| = \sqrt{1 + 9t^4}$$

$$\overrightarrow{\mathbf{T}}(t) = \frac{1}{\sqrt{1 + 9t^4}} \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}$$

$$\overrightarrow{\kappa}(t) = \frac{1}{\|\overrightarrow{\mathbf{x}}'(t)\|}\frac{d}{dt}\overrightarrow{\mathbf{T}}(t) = \frac{6t}{(1+9t^4)^2}\begin{pmatrix} 3t^2 \\ 1 \end{pmatrix}$$

$$\kappa(t) = \|\overrightarrow{\kappa}(t)\| = \frac{6|t|}{(1+9t^4)^2}\sqrt{1+9t^4} = \frac{6|t|}{(1+9t^4)^{3/2}}$$

$$\kappa(0) = 0$$

$$\lim_{t\to\infty}\kappa(t) = 0$$

Chapter 3: Functions of more than one variable

2.* (a) $x + y - z = 1$ or $z = x + y - 1$ (b) 1 (c) $(1, 1, -1)$

3. (a) $x + y + z = 3$ or $z = -x - y + 3$ (b)* $x + y = 3$ or $y = -x + 3$

4.* (1) definite (2) indefinite (3) semidefinite

5.* (a) $f(r, \theta) = \sin\theta$ (b) $f(x, y) = \frac{y}{x}\sqrt{x^2 + y^2}$

Chapter 4: Derivatives

1. Find the partial derivatives f_x and f_y of the following functions.

(1)

$$f_x(x, y) = 3(x - y)^2$$
$$f_y(x, y) = -3(x - y)^2$$

(2)

$$f_x(x, y) = -2xe^{-x^2-y^2}$$
$$f_y(x, y) = -2ye^{-x^2-y^2}$$

(3)

$$f_x(x, y) = \frac{x}{x^2 + y^2}$$
$$f_y(x, y) = \frac{y}{x^2 + y^2}$$

(4)

$$f_x(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$
$$f_y(x, y) = \frac{-2xy}{(x^2 + y^2)^2}$$

(5)

$$f_x(x, y) = ye^{xy}(\cos(xy) - \sin(xy))$$
$$f_y(x, y) = xe^{xy}(\cos(xy) - \sin(xy))$$

(6)

$$f_x(x,y) = \frac{-y}{x^2+y^2}$$
$$f_y(x,y) = \frac{x}{x^2+y^2}$$

2. (2) Note that

$$\vec{\nabla} f(x,y) = (2xy + y^2, x^2 + 2xy).$$

At the point$(1,2)$ we have

$$\vec{\nabla} f(1,2) = (8,5).$$

Therefore, when (x,y) is near the point $(1,2)$,

$$f(x,y) \approx f(1,2) + \vec{\nabla} f(1,2) \cdot (\vec{x} - \vec{x_0}),$$

that is,

$$\boxed{f(x,y) \approx 6 + 8(x-1) + 5(y-2)}.$$

(1)* From the linear approximation, the tangent plane is given by the equation

$$\boxed{z = 6 + 8(x-1) + 5(y-2)}.$$

(3) Since the gradient of f is always normal to the level curve of f, one can take the vector to be

$$\vec{\nabla} f(1,2) = \boxed{(8,5)}.$$

(4) The tangent line to the level curve of $f(x,y)$ at $(1,2)$ can be given by the level curve of the tangent plane to the graph of f at $(1,2,6)$ (the last component is $f(1,2)=6$), that is

$$z = 6 + 8(x-1) + 5(y-2) = 6.$$

Simplifying this gives

$$\boxed{8(x-1) + 5(y-2) = 0}.$$

(5) One needs to find a vector (a,b) which is normal to the gradient of f at $(1,2)$, i.e.

$$(a,b) \cdot \vec{\nabla} f(1,2) = 0,$$

or, in other words,

$$8a + 5b = 0.$$

One can choose, for example, $(a,b) = \boxed{(-5,8)}$.

3. (a) Let

$$f(x,y,z) = x^2 - 2y^2 + 3z^2.$$

Then the surface is the level surface of $f(x,y,z)$ at $(1,-1,1)$ (corresponding to level $f=2$). Since $\vec{\nabla} f(1,-1,1)$ is normal to this level surface at $(1,-1,1)$, it is also normal to the tangent plane to the level surface at $(1,-1,1)$. On the other hand, the tangent plane contains the point $\vec{x_0} = (1,-1,1)$, so it is given by the equation

$$\vec{\nabla} f(1,-1,1) \cdot (\vec{x} - \vec{x_0}) = 0.$$

Now compute

$$\vec{\nabla} f(x,y,z) = (2x, -4y, 6z),$$
$$\vec{\nabla} f(1,-1,1) = (2,4,6).$$

Therefore the equation is

$$\boxed{2(x-1) + 4(y+1) + 6(z-1) = 0}.$$

4.* Plug in $y = f(x)$ to the equation and differentiate both sides in x. By the (one-variable) chain rule, one gets

$$2x - 2f(x)f'(x) + 3f(x)^2 f'(x) = 0$$

At the point $(x,y) = (0,1)$, one has $f(0) = y = 1$, and so

$$-2f'(0) + 3f'(0) = 0$$

i.e.

$$\boxed{f'(0) = 0}.$$

At the point $(x,y) = (\sqrt{2}, -1)$, one gets

$$2\sqrt{2} + 2f'(\sqrt{2}) + 3f'(\sqrt{2}) = 0.$$

So

$$\boxed{f'(\sqrt{2}) = \frac{-2\sqrt{2}}{5}}.$$

6. (1)

$$f_{xx}(x,y) = e^x \cos y$$
$$f_{xy}(x,y) = -e^x \sin y$$
$$f_{yy}(x,y) = -e^x \cos y$$

(2)

$$f_{xx}(x,y) = \frac{2(y^2 - x^2)}{(x^2+y^2)^2}$$
$$f_{xy}(x,y) = \frac{-4xy}{(x^2+y^2)^2}$$
$$f_{yy}(x,y) = \frac{2(x^2 - y^2)}{(x^2+y^2)^2}$$

(3)

$$f_{xx}(x,y) = \frac{2xy}{(x^2+y^2)^2}$$

$$f_{xy}(x,y) = \frac{y^2 - x^2}{(x^2+y^2)^2}$$
$$f_{yy}(x,y) = \frac{-2xy}{(x^2+y^2)^2}$$

(4)*

$$f_{xx}(x,y) = \frac{2y(3x^2 - y^2)}{(x^2+y^2)^3}$$
$$f_{xy}(x,y) = \frac{2x(x^2 - 3y^2)}{(x^2+y^2)^3}$$
$$f_{yy}(x,y) = \frac{2y(y^2 - 3x^2)}{(x^2+y^2)^3}$$

7. (1) Does not exist.

(2)

$$f(x) = \boxed{\frac{1}{2}x^2y^2 + C}$$

(3)

$$f(x) = \boxed{e^{xy} + C}$$