Math 234 Review

Chapter 5: Maxima and Minima

1. (a) Solving

$$f_x = 4x + 6y + 6 = 0$$
$$f_y = 6x + 10y + 10 = 0$$

gives a unique critical point

$$(x, y) = \boxed{(0, -1)}$$

The second derivative matrix at $(0, -1)$ is

$$H = \begin{bmatrix} f_{xx}(0,-1) & f_{xy}(0,-1) \\ f_{yx}(0,-1) & f_{yy}(0,-1) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix}$$

Since $\det(H) = 40 - 36 > 0$ and $f_{xx} > 0$, by the second derivative test $(0, -1)$ is a **local minimum** point.

1. (b) Solving

$$f_x = 2x + 3y - 1 = 0$$
$$f_y = 3x + 4y = 0$$

gives a unique critical point

$$(x, y) = \boxed{(-4, 3)}$$

The second derivative matrix at $(-4, 3)$ is

$$H = \begin{bmatrix} f_{xx}(-4,3) & f_{xy}(-4,3) \\ f_{yx}(-4,3) & f_{yy}(-4,3) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Since $\det(H) = 8 - 9 < 0$, by the second derivative test $(-4, 3)$ is a **saddle** point.

1. (c) Solving

$$f_x = 6x + 6x^2 = 6x(1 + x) = 0$$
$$f_y = 4y + 4y^3 = 4y(1 + y^2) = 0$$

gives $x = 0, -1$ and $y = 0$. So there are 2 critical points:

$$\boxed{(0, 0)} \text{ and } \boxed{(-1, 0)}$$

At $(0, 0)$, the second derivative matrix is

$$\begin{bmatrix} 6 + 12x & 0 \\ 0 & 4 + 12y^2 \end{bmatrix}_{\substack{x=0 \\ y=0}} = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$$

Since $\det(H) = 24 > 0$ and $f_{xx} > 0$, by the second derivative test $(0, 0)$ is a **local minimum** point.

At $(-1, 0)$, the second derivative matrix is

$$\begin{bmatrix} 6+12x & 0 \\ 0 & 4+12y^2 \end{bmatrix}_{\substack{x=-1 \\ y=0}} = \begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix}$$

Since $\det(H) = -24 < 0$, by the second derivative test $(-1, 0)$ is a **saddle** point.

1. (d) Note that

$$f(x, y) = 3x^2y + 3xy^2 + xy$$

Now set

$$\begin{aligned} f_x &= 6xy + 3y^2 + y = y(6x + 3y + 1) = 0 \\ f_y &= 3x^2 + 6xy + x = x(3x + 6y + 1) = 0 \end{aligned}$$

If $x \neq 0$ and $y \neq 0$, we get

$$\begin{aligned} 6x + 3y + 1 = 0 \\ 3x + 6y + 1 = 0 \end{aligned}$$

Solving this gives a critical point

$$(x, y) = \boxed{(-\frac{1}{9}, -\frac{1}{9})}$$

If $x \neq 0$ and $y = 0$, we get

$$\begin{aligned} y = 0 \\ 3x + 6y + 1 = 0 \end{aligned}$$

Solving this gives a critical point

$$(x, y) = \boxed{(-\frac{1}{3}, 0)}$$

If $x = 0$ and $y \neq 0$, we get

$$\begin{aligned} 6x + 3y + 1 = 0 \\ x = 0 \end{aligned}$$

Solving this gives a critical point

$$(x, y) = \boxed{(0, -\frac{1}{3})}$$

Finally, if $x = 0$ and $y = 0$, then the equations are satisfied and we get the last critical point

$$(x, y) = \boxed{(0, 0)}$$

At $(-1/9, -1/9)$, the second derivative matrix is

$$\begin{bmatrix} 6y & 6x + 6y + 1 \\ 6x + 6y + 1 & 6x \end{bmatrix}_{\substack{x=-1/9 \\ y=-1/9}} = \begin{bmatrix} -2/3 & -1/3 \\ -1/3 & -2/3 \end{bmatrix}$$

Since $\det(H) = 1/3 > 0$ and $f_{xx} < 0$, by the second derivative test $(-1/9, -1/9)$ is a **local maximum** point.

At $(-1/3, 0)$, the second derivative matrix is

$$\begin{bmatrix} 6y & 6x + 6y + 1 \\ 6x + 6y + 1 & 6x \end{bmatrix}_{\substack{x=-1/3 \\ y=0}} = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$$

Since $\det(H) = -1 < 0$, by the second derivative test $(-1/3, 0)$ is a **saddle** point. Similarly, $(0, -1/3)$ is a **saddle** point.

At $(0, 0)$, the second derivative matrix is

$$\begin{bmatrix} 6y & 6x + 6y + 1 \\ 6x + 6y + 1 & 6x \end{bmatrix}_{\substack{x=0 \\ y=0}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since $\det(H) = -1 < 0$, by the second derivative test $(0, 0)$ is an **saddle** point.

1. (e) Solving

$$f_x = 4x^3 - 4y = 0$$
$$f_y = 4y^3 - 4x = 0$$

gives $x^9 - x = x(x^8 - 1) = 0$. Therefore $x = 0, \pm 1$ and $y = x^3$. So there are 3 critical points:

$$\boxed{(0, 0)}, \boxed{(1, 1)} \text{ and } \boxed{(-1, -1)}$$

At $(0, 0)$, the second derivative matrix is

$$\begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}_{\substack{x=0 \\ y=0}} = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

Since $\det(H) = -16 < 0$, by the second derivative test $(0, 0)$ is a **saddle** point.

At $(1, 1)$, the second derivative matrix is

$$\begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}_{\substack{x=0 \\ y=0}} = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

Since $\det(H) > 0$ and $f_{xx} > 0$, by the second derivative test $(1, 1)$ is a **local minimum** point. Similarly, $(-1, -1)$ is a **local minimum** point.

2. To find the critical points, compute and set the partial derivatives equal to 0.

$$f_x = 2x + 2(x - y) = 4x - 2y = 0 \tag{1}$$
$$f_y = -2(x - y) + 2(y - z) = -2x + 4y - 2z = 0 \tag{2}$$
$$f_z = -2(y - z) + 2(z - 1) = -2y + 4z - 2 = 0 \tag{3}$$

From (1), we get

$$y = 2x$$

Plug this into (2), we get

$$z = 3x$$

Plug these two into (3), we get

$$8x = 2$$

Therefore $x = 1/4$, and so

$$(x, y, z) = \boxed{(\frac{1}{4}, \frac{2}{4}, \frac{3}{4})}$$

is the unique critical point.

3. Suppose the vertex of the rectangular box in the first octant is (x, y, z). Then the volume of the box is given by $(2x)(2y)(2z) = 8xyz$, where (x, y, z) is restricted to the given surface. So we need to maximize

$$f = 8xyz$$

subject to the constraint

$$g = x^4 + y^4 + z^4 - 1 = 0$$

Using the method of Lagrange multiplier, we need to solve

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$f_z = \lambda g_z$$
$$g = 0$$

i.e.

$$8yz = \lambda 4x^3 \tag{4}$$
$$8xz = \lambda 4y^3 \tag{5}$$
$$8xy = \lambda 4z^3 \tag{6}$$
$$x^4 + y^4 + z^4 = 1 \tag{7}$$

Multiply (4) by x, (5) by y, (6) by z, we get

$$\begin{aligned} 8xyz &= \lambda 4x^4 \\ 8xyz &= \lambda 4y^4 \\ 8xyz &= \lambda 4z^4 \end{aligned}$$

Since the left hand sides are all equal, we get

$$\lambda 4x^4 = \lambda 4y^4 = \lambda 4z^4$$

Note that $\lambda \neq 0$, since otherwise by (4) we get $yz = 0$, contradicting the fact that the maximum volume (supposedly exists) $8xyz$ should be positive. Now we can cancel out $\lambda 4$ and get

$$x^4 = y^4 = z^4$$

Since x, y, z are all positive, this implies

$$x = y = z$$

Finally, plug this into (7), we get

$$3x^4 = 1$$

i.e.

$$x = \frac{1}{\sqrt[4]{3}}$$

So to maximize the volume, we need

$$\boxed{x = y = z = \frac{1}{\sqrt[4]{3}}}$$

and the maximum volume is

$$\boxed{\frac{8}{3^{3/4}}}$$

4. Let (x, y, z) be a point on the surface, then its distance to the origin is

$$\sqrt{x^2 + y^2 + z^2}$$

So we need to minimize $\sqrt{x^2 + y^2 + z^2}$ subject to the constraint

$$g = x^2 + 4y^2 + 9z^2 - 36 = 0$$

For convenience we may instead minimize the squared distance

$$f = x^2 + y^2 + z^2$$

By the method of Lagrange multiplier, we need to solve

$$2x = \lambda 2x \tag{8}$$
$$2y = \lambda 8y \tag{9}$$
$$2z = \lambda 18z \tag{10}$$
$$x^2 + 4y^2 + 9z^2 = 36 \tag{11}$$

If $x \neq 0$, then one can cancel out $2x$ in (8) and get $\lambda = 1$. But then (9) and (10) imply $y = 0$ and $z = 0$. And (11) in turn implies $x^2 = 36$, that is, $x = \pm 6$. So in this case we get two critical points: $(\pm 6, 0, 0)$.

If $x = 0$ and $y \neq 0$, then one can cancel out $2y$ in (8) and get $\lambda = 1/4$. But then (10) implies $z = 0$, and (11) in turn implies $4y^2 = 36$, that is, $y = \pm 3$. So in this case we also get two critical points: $(0, \pm 3, 0)$.

Finally, if $x = 0$ and $y = 0$, then (11) implies $9z^2 = 36$, that is, $z = \pm 2$. So we get two more critical points: $(0, 0, \pm 2)$.

Now by testing all the 6 critical points we see that $f(0,0,2) = f(0,0,-2) = 4$ has the smallest value. So $\boxed{(0, 0, \pm 2)}$ are the closest points to the origin. And the closest distance is $\boxed{2}$.

5.* It seems tricky to solve this problem using the method of Lagrange multiplier. Here we give a solution by reducing it to a one-variable optimization problem.

Suppose the cup has height h and the base has radius r. Then the area of the base is πr^2, and the area of the side is $2\pi r h$. Since the density is 1 per unit area, the total mass is

$$\pi r^2 + 2\pi r h = S \tag{12}$$

The capacity (i.e. volume) we need to maximize is

$$f = \pi r^2 h = r \cdot \pi r h \tag{13}$$

From (12) we get

$$\pi r h = \frac{S - \pi r^2}{2}$$

Plug this into (13) we get

$$f(r) = \frac{1}{2}(Sr - \pi r^3)$$

where $r > 0$ and $\pi r^2 < S$. So we only need to maximize the one-variable function $f(r)$ in its domain.

Now differentiate and set

$$f'(r) = \frac{1}{2}(S - 3\pi r^2) = 0$$

Since r is positive we get

$$\boxed{r = \sqrt{\frac{S}{3\pi}}}$$

By the first derivative test this is the global maximum of $f(r)$ on the positive real line. So the maximum volume is

$$\frac{1}{2}\sqrt{\frac{S}{3\pi}}\left(S - \pi\frac{S}{3\pi}\right) = \boxed{\frac{1}{3\sqrt{3\pi}} S^{3/2}}$$

Chapter 6: Integrals

1. (a) One can rewrite the domain D as

$$D = \{(x, y) : 0 \le x \le 1, 0 \le y \le x\}$$

Therefore

$$\begin{aligned}
\iint_D \sin(x^2) dA &= \int_0^1 \int_0^x \sin(x^2) dy dx \\
&= \int_0^1 x \sin(x^2) dx \\
&= \frac{1}{2} \int_0^1 \sin(u) du \qquad (u = x^2) \\
&= \frac{1}{2} \big[-\cos(u) \big]_0^1 \\
&= \boxed{\frac{1 - \cos(1)}{2}}
\end{aligned}$$

1. (b) One can rewrite the domain D as

$$D = \{(x, y) : 0 \le x \le 1, 0 \le y \le \sqrt{x}\}$$

Therefore

$$\begin{aligned}
\iint_D \frac{y}{1 + x^2} dA &= \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1 + x^2} dy dx \\
&= \int_0^1 \frac{1}{1 + x^2} \Big(\int_0^{\sqrt{x}} y dy \Big) dx \\
&= \frac{1}{2} \int_0^1 \frac{x}{1 + x^2} dx \\
&= \frac{1}{4} \int_1^2 \frac{1}{u} du \qquad (u = x^2 + 1) \\
&= \frac{1}{4} \big[\ln u \big]_1^2 \\
&= \boxed{\frac{\ln 2}{4}}
\end{aligned}$$

1. (c) Using the given order, we have

$$\begin{aligned}
\iint_D xe^y dA &= \int_0^1 \int_{\sqrt{y}}^{y} xe^y dx dy \\
&= \frac{1}{2}\int_0^1 e^y(y^2 - y)dy \\
&= \frac{1}{2}\int_0^1 y^2 e^y dy - \frac{1}{2}\int_0^1 ye^y dy \\
&= \frac{1}{2}\int_0^1 y^2 de^y - \frac{1}{2}\int_0^1 ye^y dy \\
&= \frac{1}{2}\Big(y^2e^y\Big|_0^1 - 2\int_0^1 ye^y dy\Big) - \frac{1}{2}\int_0^1 ye^y dy \quad \text{(integrate by parts)} \\
&= \frac{e}{2} - \frac{3}{2}\int_0^1 ye^y dy \\
&= \frac{e}{2} - \frac{3}{2}\Big(ye^y\Big|_0^1 - \int_0^1 e^y dy\Big) \quad \text{(integrate by parts)} \\
&= \frac{e}{2} - \frac{3}{2}\Big(e - (e-1)\Big) \\
&= \boxed{\frac{e}{2} - \frac{3}{2}}
\end{aligned}$$

1. (d) Using the polar coordinates, we can write the domain D as

$$D = \{(x, y) : 0 \le \theta \le 2\pi, 0 \le r \le \sqrt{2}\}$$

Therefore

$$\begin{aligned}
\iint_D \sqrt{4 - x^2 - y^2} dA &= \int_0^{2\pi} \Big(\int_0^{\sqrt{2}} \sqrt{4 - r^2}\, r dr\Big) d\theta \\
&= \int_0^{2\pi} \Big(-\frac{1}{2}\int_4^2 \sqrt{u} du\Big) d\theta \quad (u = 4 - r^2) \\
&= \int_0^{2\pi} \Big(-\frac{1}{2}\Big[\frac{2}{3}u^{3/2}\Big]_4^2\Big) d\theta \\
&= \int_0^{2\pi} \frac{1}{3}\Big(4^{3/2} - 2^{3/2}\Big) d\theta \\
&= \boxed{\frac{4\pi}{3}\Big(4 - \sqrt{2}\Big)}
\end{aligned}$$

2. (a) This is the region under the graph $z = y$ over the domain D, where D is bounded by $y^2 = 4 - x$, $y^2 = x$ in the first quadrant. Alternatively,

$$D = \{(x, y) : 0 \le y \le \sqrt{2}, y^2 \le x \le 4 - y^2\}$$

Therefore the volume equals

$$\begin{aligned}
\iint_D y dA &= \int_0^{\sqrt{2}} \int_{y^2}^{4-y^2} y dx dy \\
&= \int_0^{\sqrt{2}} y(4 - y^2 - y^2) dy \\
&= \int_0^{\sqrt{2}} (4y - 2y^3) dy \\
&= \left[2y^2 - \frac{y^4}{2}\right]_0^{\sqrt{2}} \\
&= 4 - 2 \\
&= \boxed{2}
\end{aligned}$$

2. (b) This is the region between the graphs $z = \sqrt{x^2 + y^2}$ and $z = x^2 + y^2$ over the domain D, where D is bounded by the intersection of the two graphs, and is restricted to the first quadrant. More precisely,

$$D = \{(x, y) : x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$$

Therefore, using the polar coordinates, the volume equals

$$\begin{aligned}
\iint_D \sqrt{x^2 + y^2} - (x^2 + y^2) dA &= \int_0^{\pi/2} \int_0^1 (r - r^2) r dr d\theta \\
&= \int_0^{\pi/2} \int_0^1 (r^2 - r^3) dr d\theta \\
&= \int_0^{\pi/2} (\frac{1}{3} - \frac{1}{4}) d\theta \\
&= \boxed{\frac{\pi}{24}}
\end{aligned}$$