Math 234 Review

Chapter 6: Moment of Inertia, Cylindrical and Spherical Coordinates

1. (*Moment of inertia will not be on the exam. The computation here is a practice of using the cylindrical coordinates.*) By equation (138) on page 129, we need to compute the triple integral

$$M = \iiint_D \mu r^2 dV.$$

Writing this in the cylindrical coordinates, we get

$$\begin{aligned} M &= \int_0^{2\pi}\int_0^{100}\int_0^{100} \frac{1}{x^2+y^2} e^{-x^2-y^2-z} r^2 dz r dr d\theta \\ &= \int_0^{2\pi}\int_0^{100}\int_0^{100} \frac{1}{r^2} e^{-r^2-z} r^3 dz dr d\theta \\ &= \int_0^{2\pi}\int_0^{100}\int_0^{100} r e^{-r^2} e^{-z} dz dr d\theta \\ &= (1-e^{-100})\int_0^{2\pi}\int_0^{100} r e^{-r^2} dr d\theta \\ &= (1-e^{-100})\int_0^{2\pi} \frac{1}{2}(1-e^{10000}) d\theta \\ &= \boxed{(1-e^{-100})(1-e^{10000})\pi} \\ &\approx \pi \end{aligned}$$

2. (*Moment of inertia will not be on the exam. The computation is a practice of using the spherical coordinates.*) By equation (138) on page 129, we need to compute the triple integral

$$M = \iiint_D \mu r^2 dV.$$

Writing this in the spherical coordinates, we get

$$\begin{aligned} M &= \int_0^1\int_0^{2\pi}\int_0^{\pi/4} r^2 \cdot \rho^2 \sin\phi d\phi d\theta d\rho \\ &= \int_0^1\int_0^{2\pi}\int_0^{\pi/4} \rho^2 \sin^2\phi \cdot \rho^2 \sin\phi d\phi d\theta d\rho \quad (r^2 = x^2+y^2 = \rho^2\sin^2\phi) \\ &= \int_0^1\int_0^{2\pi}\int_0^{\pi/4} \rho^4 \sin^3\phi d\phi d\theta d\rho \\ &= \int_0^1\int_0^{2\pi} \rho^4 \int_0^{\pi/4} (1-\cos^2\phi)\sin\phi d\phi d\theta d\rho \\ &= \int_0^1\int_0^{2\pi} \rho^4 \int_0^{\pi/4} (\sin\phi - \cos^2\phi\sin\phi) d\phi d\theta d\rho \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_0^{2\pi} \rho^4 \Big\{ [-\cos\phi]_0^{\pi/4} + \frac{1}{3}[\cos^3\phi]_0^{\pi/4} \Big\} d\theta d\rho \\
&= \int_0^1 \int_0^{2\pi} \rho^4 \Big\{ -\frac{1}{\sqrt{2}} + 1 + \frac{1}{3}(\frac{1}{2\sqrt{2}} - 1) \Big\} d\theta d\rho \\
&= \frac{1}{5}\Big(\frac{2}{3} - \frac{5}{6\sqrt{2}}\Big) 2\pi \\
&= \boxed{\Big(\frac{4}{15} - \frac{1}{3\sqrt{2}}\Big)\pi}
\end{aligned}
$$

Chapter 7: Vector Calculus

1. According to page 141, the average of f on $\mathcal{C}$ is given by

$$\frac{\int_{\mathcal{C}} f ds}{\int_{\mathcal{C}} ds}.$$

In our case

$$f = \cos\theta = \frac{\overrightarrow{\mathbf{x}} \cdot \overrightarrow{\mathbf{e}_3}}{\|\overrightarrow{\mathbf{x}}\| \|\overrightarrow{\mathbf{e}_3}\|} = \frac{t}{\sqrt{1+t^2}}.$$

On the other hand,

$$ds = \|\overrightarrow{\mathbf{x}}'(t)\| dt = \sqrt{1+t^2} dt.$$

So

$$\boxed{\int_{\mathcal{C}} f ds} = \int_0^{1000} \frac{t}{\sqrt{1+t^2}} \sqrt{1+t^2} dt = \int_0^{1000} t dt = \boxed{500000}.$$

Computing $\int_{\mathcal{C}} ds$ involves the integral $\int \sqrt{1+t^2}\, dt$, which will not be required on the test. (It turns out $\int_{\mathcal{C}} ds \approx 500004$. Therefore the average is about 1.)

2. Since the domain D is symmetric about x and y, it suffices to compute

$$\overline{x} = \frac{\int_{\mathcal{C}} x ds}{\int_{\mathcal{C}} ds}.$$

Note that $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$, where

$$
\begin{aligned}
&\mathcal{C}_1 : x = t, y = 0, 0 \le t \le 1 \\
&\mathcal{C}_2 : x = 0, y = t, 0 \le t \le 1 \\
&\mathcal{C}_3 : x = \cos t, y = \sin t, 0 \le t \le \pi/2.
\end{aligned}
$$

In particular, the denominator $\int_{\mathcal{C}} ds$ equals the arclength of $\mathcal{C}$, which is $1 + 1 + \pi/2 = 2 + \frac{\pi}{2}$. To compute the numerator, we write

$$\int_{\mathcal{C}} x ds = \int_{\mathcal{C}_1} x ds + \int_{\mathcal{C}_2} x ds + \int_{\mathcal{C}_3} x ds.$$

Where

$$\int_{\mathcal{C}_1} x ds = \int_0^1 t dt = 1/2,$$

$$\int_{\mathcal{C}_2} x ds = \int_0^1 0 dt = 0,$$

and

$$\int_{\mathcal{C}_3} x ds = \int_0^{\pi/2} \cos t dt = 1 \text{ (note that } \|\vec{\mathbf{x}}'(t)\| = 1).$$

So

$$\int_{\mathcal{C}} x ds = \frac{1}{2} + 0 + 1 = \frac{3}{2},$$

and therefore

$$\boxed{\overline{x} = \overline{y}} = \frac{3/2}{2+\pi/2} = \boxed{\frac{3}{4+\pi}} \approx 0.42.$$

3. Using the given parametrization,

$$\begin{aligned}
\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} &= \int_{\mathcal{C}} x dx + y dy + z dz \\
&= \int_0^1 t dt + t^2 dt^2 + t^3 dt^3 \\
&= \int_0^1 t dt + t^2(2t)dt + t^3(3t^2)dt \\
&= \int_0^1 (t + 2t^3 + 3t^5) dt \\
&= \left(\frac{t^2}{2} + \frac{2t^4}{4} + \frac{3t^6}{6}\right)\Big|_0^1 \\
&= \boxed{\frac{3}{2}}
\end{aligned}$$

4. (1) We compute it using Green's theorem (for circulation).

$$\begin{aligned}
\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{x}} &= \int_D \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} dA \\
&= \int_D \left\{\frac{\partial(xy^2)}{\partial x} - \frac{\partial(x^2y)}{\partial y}\right\} dA \\
&= \int_0^1 \int_0^1 (y^2 - x^2) dx dy \\
&= \boxed{0}
\end{aligned}$$

(2) $\vec{\mathbf{F}}$ is *not* conservative since $Q_x \neq P_y$.

(3) Using Green's theorem (for flux),

$$\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} ds = \int_D \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \end{pmatrix} dA$$
$$= \int_D (2xy + 2xy) dA$$
$$= \int_0^1 \int_0^1 4xy dx dy$$
$$= \boxed{1}$$

5. (1) Using Green's theorem (for circulation),

$$\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds = \oint_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{x}}$$
$$= -\int_D \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} dA \quad (- \text{ sign due to clock-wise orientation})$$
$$= -\int_D (2xy - (-2xy)) dA$$
$$= -\int_0^\pi \int_1^2 4xy \cdot r dr d\theta \quad (\text{polar coordinates})$$
$$= -\int_0^\pi \int_1^2 (4r\cos\theta r \sin\theta) \cdot r dr d\theta$$
$$= -\int_0^\pi \int_1^2 4r^3 \cos\theta \sin\theta dr d\theta$$
$$= -15 \int_0^\pi \cos\theta \sin\theta d\theta$$
$$= -15 \left[\frac{\sin^2\theta}{2}\right]_0^\pi$$
$$= \boxed{0}$$

Using Green's theorem (for flux),

$$\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{N}} ds = -\int_D \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \end{pmatrix} dA \quad (- \text{ sign due to clock-wise orientation})$$
$$= -\int_D (-y^2 + x^2) dA$$
$$= -\int_0^\pi \int_1^2 (x^2 - y^2) \cdot r dr d\theta \quad (\text{polar coordinates})$$
$$= -\int_0^\pi \int_1^2 (r^2\cos^2\theta - r^2\sin^2\theta) \cdot r dr d\theta$$
$$= -\int_0^\pi \int_1^2 r^3(\cos^2\theta - \sin^2\theta) dr d\theta$$

$$
\begin{aligned}
&= -\frac{15}{4}\int_0^{\pi} (\cos^2\theta - \sin^2\theta)d\theta \\
&= -\frac{15}{4}\int_0^{\pi} \left\{\frac{1+\cos 2\theta}{2} - \frac{1-\cos 2\theta}{2}\right\}d\theta \\
&= -\frac{15}{4}\int_0^{\pi} \cos 2\theta d\theta \\
&= -\frac{15}{4}\frac{\sin 2\theta}{2}\Big|_0^{\pi} \\
&= \boxed{0}
\end{aligned}
$$

(2) $\overrightarrow{\mathbf{F}}$ is *not* conservative since $Q_x - P_y \neq 0$. $\overrightarrow{\mathbf{F}}$ is *not* divergence-free since $P_x + Q_y \neq 0$.

6. One can view this as a circulation

$$\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{x}}$$

where

$$\overrightarrow{\mathbf{F}} = \begin{pmatrix} (y^2+x)e^x \\ 0 \end{pmatrix}.$$

Using Green's theorem (for circulation),

$$
\begin{aligned}
\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{x}} &= -\int_D \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} dA \\
&= \int_D 2ye^x dA \\
&= \int_0^1 \int_{\sqrt{x}}^1 2ye^x dydx \\
&= \int_0^1 e^x(1-x)dx \\
&= \int_0^1 e^x dx - \int_0^1 xe^x dx \\
&= \int_0^1 e^x dx - \left(xe^x\Big|_0^1 - \int_0^1 e^x dx\right) \quad \text{(integration by parts)} \\
&= 2\int_0^1 e^x dx - e \\
&= \boxed{e-2}
\end{aligned}
$$