Math 276 Discussion Worksheet 1

1. *Solution.* Notice that

$$\begin{aligned} f(x) &= \int_0^{x^2} t^3 dt + \int_{-x^2}^{0} t^3 dt \\ &= \int_0^{x^2} t^3 dt - \int_0^{-x^2} t^3 dt. \end{aligned}$$

If we write

$$F(u) = \int_0^u t^3 dt,$$

then

$$f(x) = F(x^2) - F(-x^2).$$

Using the chain rule, we find

$$f'(x) = F'(x^2)(2x) - F'(-x^2)(-2x).$$

But

$$F'(u) = u^3,$$

so

$$f'(x) = (x^2)^3(2x) - (-x^2)^3(-2x) = 0.$$

2. *Solution.* Notice that

$$\begin{aligned} F(x) &= \int_0^x xf(t) - tf(t)dt \\ &= \int_0^x xf(t)dt - \int_0^x tf(t)dt \\ &= x\Big(\int_0^x f(t)dt\Big) - \int_0^x tf(t)dt. \end{aligned}$$

Applying the product rule to the first term, we find

$$\begin{aligned} F'(x) &= \Big(\int_0^x f(t)dt + xf(x)\Big) - xf(x) \\ &= \int_0^x f(t)dt. \end{aligned}$$

Therefore, F' is differentiable and

$$F''(x) = f(x).$$

3. *Solution.* Prove by induction on n. If $n = 0$, then we need to show that $f = a_0$ if and only if $f' = 0$. But this is clearly true. Suppose that the statement holds for $n - 1$, we now show that it also holds for n.

Indeed, it is easy to see that f is of the form

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

if and only if f' is of the form

$$f'(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}. \tag{1}$$

Note that f' is now n times differentiable. Thus by the induction hypothesis, (1) holds if and only if $(f')^{(n)} = 0$, in other words,

$$f^{(n+1)} = 0.$$

This finishes the proof.