Math 276 Discussion Worksheet 4

1. (i) Use the identity

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \frac{x^{n+1}}{1-x}, \ x \neq 1$$

to show that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots + (-1)^n x^{2n} + (-1)^{n+1} \frac{x^{2n+2}}{1+x^2}.$$

(ii) Use the identity

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt$$

to show that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + R_{2n+1}(x)$$

where

$$|R_{2n+1}(x)| \leq \frac{|x|^{2n+3}}{2n+3}.$$

(iii) Use (ii) to show that (at least formally)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

2. Let p be a positive number. (i) If k is a positive integer, show that

$$\int_{k-1}^{k} t^p dt \leq k^p \leq \int_{k}^{k+1} t^p dt.$$

(ii) Define

$$S_p(n) = 1^p + 2^p + \cdots + n^p.$$

Use (i) to show that

$$\frac{n^{p+1}}{p+1} \leq S_p(n) \leq \frac{(n+1)^{p+1}}{p+1}.$$

(iii) Use (ii) to show that

$$\lim_{n\to\infty} \frac{S_p(n)}{n^{p+1}} = \frac{1}{p+1}.$$