1. To find the critical numbers, set

$$f'(x) = 6x^2 - 6x = 0$$

that is,

$$6x(x-1) = 0.$$

Therefore $x = 0$ and $x = 1$ are the critical numbers. The corresponding critical points are $(0, 4)$ and $(1, 3)$. Next, the endpoints are $(-1, -1)$ and $(2, 8)$.

So the maximum of f is $\max\{4, 3, -1, 8\} = 8$, attained at $x = 2$; the minimum of f is $\max\{4, 3, -1, 8\} = -1$, attained at $x = -1$.

2. We need to find c in $(-2, 2)$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)}.$$

One computes that $f(2) = 4$, $f(-2) = 0$, and

$$f'(c) = 3c^2 - 3.$$

Therefore we need

$$3c^2 - 3 = 1.$$

Solving this gives

$$c^2 = \frac{4}{3}$$

that is

$$c = \pm\frac{2}{\sqrt{3}}$$

which are both in $(-2, 2)$.

Bonus. Following the hint, to show that $f(x) > 0$ for any fixed $x > 0$, we write

$$f(x) = f(x) - f(0)$$

using $f(0) = 0$. By the MVT (with $a = 0$ and $b = x$),

$$f(x) - f(0) = f'(c)(x - 0)$$

for some c in $(0, x)$. On the other hand, $f'(c) = 1 - \cos(c) \geq 0$. Therefore

$$f'(c)(x - 0) = (1 - \cos(c))x \geq 0.$$

This inequality is strict if $0 < x \leq 2\pi$, since

$$\cos(c) < 1$$

for any c in $(0, 2\pi)$. If $c > 2\pi$, we can write

$$f(x) = [f(x) - f(2\pi)] + f(2\pi).$$

By the same argument as above
$$f(x) - f(2\pi) \geq 0.$$
On the other hand, taking $x = 2\pi$ we see that $f(2\pi) > 0$. Therefore
$$[f(x) - f(2\pi)] + f(2\pi) > 0.$$
This finishes the proof.