Theorem 2.6.4 (Bolzano-Weierstrass Theorem) *If the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded, then it has a convergent subsequence.*

Proof. (*Some pictures of the proof can be found* here, *where $\{a_n\}$ is denoted $\{x_n\}$.*) Since the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded, there exists $M > 0$ such that

$$|a_n| \leq M, \ \forall n \geq 1.$$

In other words,

$$a_n \in [-M, M], \ \forall n \geq 1. \text{ (the symbel } \in \text{ means 'belongs to')}$$

We now describe how we can find inductively a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$.

First, let $n_1 = 1$ and $I_1 = [-M, M]$. Suppose n_k and I_k have been chosen, we now proceed to choose n_{k+1} and I_{k+1}. Divide I_k into two intervals of equal length, denoted J_1 and J_2. Consider two cases:

Case 1: If there are infinitely many indexes n such that $a_n \in J_1$, then we can find one such n with $n > n_k$ (since there are infinitely many of them). Take n_{k+1} to be this n. Take I_{k+1} to be J_1. This finishes our choice of n_{k+1} and I_{k+1}.

Case 2: If otherwise there only finitely many indexes n such that $a_n \in J_1$, then there must be infinitely many indexes n such that $a_n \in J_2$. In particular we can find one such n with $n > n_k$. In this case, take n_{k+1} to be this n, and take I_{k+1} to be J_2.

By induction we obtain a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ which satisfies

$$a_{n_k} \in I_k, \ \forall k \geq 1.$$

Observe also that the intervals I_k satisfy

$$I_{k+1} \subset I_k \text{ (nested; the symbel } \subset \text{ means 'is contained in')}$$

and

$$|I_k| = \frac{2M}{2^{k-1}} \text{ (shrinking; here } |I_k| \text{ denotes the length of } I_k\text{).}$$

To show that $\{a_{n_k}\}_{k=1}^{\infty}$ is convergent, by Theorem 2.5.9 it remains/suffices to show that $\{a_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence. To this end, let $\varepsilon > 0$ be any given positive number. Since

$$\lim_{k\to\infty} \frac{2M}{2^{k-1}} = 0,$$

there exists k^* such that

$$\frac{2M}{2^{k^*-1}} < \varepsilon.$$

For all $k \geq k^*$ and $\ell \geq k^*$, by the nesting property of the intervals I_k, we have

$$a_{n_k} \in I_k \subset I_{k-1} \subset \cdots \subset I_{k^*}$$

and

$$a_{n_\ell} \in I_\ell \subset I_{\ell-1} \subset \cdots \subset I_{k^*}.$$

In particular, both a_{n_k} and a_{n_ℓ} are contained in I_{k^*}. From this we obtain

$$|a_{n_k} - a_{n_\ell}| \leq |I_{k^*}|.$$

On the other hand, by our choice of k^*,

$$|I_{k^*}| = \frac{2M}{2^{k^*-1}} < \varepsilon.$$

Combining these, we get

$$|a_{n_k} - a_{n_\ell}| < \varepsilon, \ \forall k, \ell \geq k^*.$$

By definition, this shows $\{a_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence, and the proof is complete. □