5.3#6(a) ($\Rightarrow$) Suppose $f'(c) \geq 0$ for all $c \in (a,b)$. Then for any $x, y \in (a,b)$ with $x < y$, by the Mean Value Theorem we have, for some $c \in (x,y)$

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

or, equivalently,

$$f(y) - f(x) = f'(c)(y - x).$$

Since $f'(c) \geq 0$ and $y - x > 0$, it follows that $f(y) - f(x) \geq 0$, that is, $f(y) \geq f(x)$. This shows f is increasing, since $f(y) \geq f(x)$ whenever $x > y$.
($\Leftarrow$) Suppose f is increasing on (a,b). We show that $f'(x) \geq 0$ for all $x \in (a,b)$. By the definition of derivative, we have

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x}.$$

On the other hand, for any $x < y < b$, since f is increasing, we have $f(y) - f(x) \geq 0$. Therefore

$$\frac{f(y) - f(x)}{y - x} \geq 0$$

for all $y \in (x,b)$. Taking the limit as $y \to x^+$, we get $f'(x) \geq 0$, as desired.

5.3#15(e) Take $f(x) = \sin(x)$. Note that f is differentiable on $(-\infty, \infty)$ and $f'(x) = \cos(x)$. Suppose $x < y$. By the Mean Value Theorem, we have for some $c \in (x,y)$,

$$f(y) - f(x) = f'(c)(x - y).$$

Therefore

$$|\sin(y) - \sin(x)| = |\cos(c)||x - y|.$$

But $|\cos(c)| \leq 1$. So we obtain

$$|\sin(y) - \sin(x)| \leq |x - y|.$$

If $x > y$, then reverting x and y in the argument above gives that same bound. Finally, if $x = y$, then trivially

$$|\sin(y) - \sin(x)| = 0 \leq |x - y|.$$

Therefore $|\sin(y) - \sin(x)| \leq |x - y|$ holds for all x and y.

For the second part of the question, notice that since $\sin(x)$ is an odd function, we have $\sin(x) = -\sin(-x)$. Therefore, applying the inequality proven above, we get

$$|\sin(y) + \sin(x)| = |\sin(y) - \sin(-x)| \leq |y - (-x)| = |y + x|.$$

This shows

$$|\sin(y) + \sin(x)| \leq |y + x|.$$

5.4#9 Suppose $f''(x) = 0$. Then by Corollary 5.3.7(a), we have

$$f'(x) = a$$

for some constant a. Let $g(x) = ax$, then $f'(x) = g'(x)$ for all x. So by Corollary 5.3.7(b), we must have

$$f(x) = g(x) + b$$

for some constant b. This shows

$$f(x) = ax + b,$$

as desired.

5.4#29 We prove the statement by induction. The case $n = 1$ is just Theorem 5.3.1 (Rolle's theorem). Suppose the statement holds for some $n \geq 1$. Applying Theorem 5.3.1 to each of the intervals $[x_{k-1}, x_k]$, $k = 1, \cdots, n+1$, we obtain $c_{k-1} \in (x_{k-1}, x_k)$, $k = 1, \cdots, n+1$ at which $f'(c_k) = 0$. Now f' is an n-times differentiable function with $n+1$ distinct zeros $c_0, \cdots, c_n$. So by the induction hypothesis there must exist c such that

$$f^{(n+1)}(c) = 0.$$

This completes the proof.