6.1#2 The proof is the same as that of Lemma 6.1.5, with m_i replaced by M_i, and '$\leq$' replaced by '$\geq$'. Note how to pass from $Q = P \cup \{c\}$ to general refinements of P.

6.1#8 Suppose $P = \{x_k\}_{k=1}^n$. Note that, since

$$f(x) + g(x) \leq \sup_{[x_{k-1},x_k]} f + \sup_{[x_{k-1},x_k]} g$$

for all $x \in [x_{k-1}, x_k]$, we have

$$\sup_{[x_{k-1},x_k]} (f+g) \leq \sup_{[x_{k-1},x_k]} f + \sup_{[x_{k-1},x_k]} g.$$

In other words,

$$M_k(f+g) \leq M_k(f) + M_k(g).$$

Therefore

$$\begin{aligned}
U(P, f+g) &= \sum_{k=1}^n M_k(f+g)\Delta x_k \\
&\leq \sum_{k=1}^n \big(M_k(f) + M_k(g)\big)\Delta x_k \\
&= \sum_{k=1}^n M_k(f)\Delta x_k + \sum_{k=1}^n M_k(g)\Delta x_k \\
&= U(P, f) + U(P, g).
\end{aligned}$$

This proves the inequality.

6.2#13 By assumption, $f(x) = 0$ except at finitely many points in $[a, b]$, say at $c_1 < \cdots < c_n$. Without loss of generality, assume that $a < c_1$ and $c_n < b$ (otherwise the proof is similar). For sufficiently small $\varepsilon > 0$, let P be the partition

$$P = \{a, c_1 \pm \varepsilon, \cdots, c_n \pm \varepsilon, b\}.$$

Since f takes value 0 outside of the intervals $(c_k - \varepsilon, c_k + \varepsilon)$, $k = 1, \cdots, n$, we have that

$$U(P, f) = \sum_{k=1}^n M_k \cdot (2\varepsilon) = 2\varepsilon \cdot \sum_{k=1}^n M_k,$$

$$L(P, f) = \sum_{k=1}^n m_k \cdot (2\varepsilon) = 2\varepsilon \cdot \sum_{k=1}^n m_k$$

where

$$M_k = \max\{f(c_k), 0\}, \quad m_k = \min\{f(c_k), 0\}.$$

Thus,

$$2\varepsilon \cdot \sum_{k=1}^n m_k \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq 2\varepsilon \cdot \sum_{k=1}^n M_k.$$

Since ε can be made arbitrarily small, it follows that

$$\underline{\int_a^b} f = \overline{\int_a^b} f = 0.$$

This shows $f \in R[a, b]$ and $\int_a^b f = 0$.