HW problems for Section 2.5

1. Suppose a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $a_n \geq 0, \forall n \geq 1$ and $\lim_{n\to\infty} a_n = A$. Show that $A \geq 0$ and $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{A}$.

2. Let $\{a_n\}_{n=1}^{\infty}$ be any given sequence. Show that the sequence $\{b_n\}_{n=1}^{\infty}$ defined by

$$b_n = \max\{a_1, \cdots, a_n\}$$

is increasing, and therefore either converges to a finite number or diverges to infinity. (*Remark*: one can show that $\lim_{n\to\infty} b_n = \sup\{a_n : n \geq 1\}$.)

3. Suppose $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Then the sequence $\{b_n\}_{n=1}^{\infty}$ defined by

$$b_n = \sup\{|a_m - a_n| : m \geq n\}$$

converges to 0. (*Remark*: the converse statement is also true.)

4*. Show that every convergent sequence can be written as a sum of an increasing sequence and a sequence that converges to 0.

Notation: Let E be a nonempty set of real numbers, then

$$\inf E := \max\{x : x \leq y, \forall y \in E\},$$

$$\sup E := \min\{x : x \geq y, \forall y \in E\}.$$