**2.1#5** Assume that $\{a_n\}_{n=1}^{\infty}$ converges to $A$. By definition for any $\varepsilon > 0$, there exists $n^*$ such that

$$|a_n - A| < \varepsilon, \ \forall n \geq n^*.$$

By the triangle inequality we have

$$||a_n| - |A|| \leq |a_n - A|.$$

So it follows that

$$||a_n| - |A|| < \varepsilon, \ \forall n \geq n^*.$$

This shows that $\{|a_n|\}_{n=1}^{\infty}$ converges to $|A|$.

The converse is false. Consider $a_n = (-1)^n$. Then $\{|a_n| = 1\}_{n=1}^{\infty}$ converges to 1. But $\{a_n = (-1)^n\}_{n=1}^{\infty}$ diverges.

**2.1#12** By Theorem 2.1.12, since $\{a_n\}_{n=1}^{\infty}$ converges to $A \neq 0$, there exists $n^*$ such that

$$|a_n| \geq \frac{|A|}{2}, \ \forall n \geq n^*.$$

From this we get

$$\frac{1}{|a_n|} \leq \frac{2}{|A|}, \ \forall n \geq n^*.$$

On the other hand, by the assumption we have $a_n \neq 0, \ \forall n \geq 1$. So we can take

$$M = \max\left\{\frac{1}{|a_1|}, \cdots, \frac{1}{|a_{n^*-1}|}, \frac{2}{|A|}\right\},$$

so that

$$\frac{1}{|a_n|} \leq M, \ \forall n \geq 1.$$

This shows $\{\frac{1}{a_n}\}_{n=1}^{\infty}$ is a bounded sequence.

**2.2#5** Since $\{b_n\}_{n=1}^{\infty}$ is bounded, there exists $M > 0$ such that

$$|b_n| \leq M, \ \forall n \geq 1.$$

It follows that

$$0 \leq |a_n b_n| \leq M|a_n|, \ \forall n \geq 1.$$

Since $\{a_n\}_{n=1}^{\infty}$ converges to 0, by Theorem 2.1.14 we have $\{|a_n|\}_{n=1}^{\infty}$ converges 0, and therefore

$$\lim_{n\to\infty} M|a_n| = 0.$$

By the squeeze theorem, it follows that

$$\lim_{n\to\infty} |a_n b_n| = 0.$$

By Theorem 2.1.14 again, we then have

$$\lim_{n\to\infty} a_n b_n = 0,$$

as desired.

**2.3#6(a)** Since $\{b_n\}_{n=1}^{\infty}$ converges to $B > 0$, by definition, with $\varepsilon = B/2$ there exists $n^*$ such that

$$|b_n - B| < B/2, \ \forall n \geq n^*.$$

Since

$$B - b_n \leq |b_n - B|,$$

it follows that

$$b_n \geq B/2, \ \forall n \geq n^*.$$

Applying Theorem 2.3.3(b) with $K = B/2$, we conclude that

$$\lim_{n\to\infty} a_n b_n = \infty,$$

as desired.