2.4#15(b) First, the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing. We show this by induction. Obviously, $a_1 = 1 \leq a_2 = \sqrt{2}$. Suppose $a_n \leq a_{n+1}$ holds. Then by the recurrence relation we have

$$a_{n+2} = \sqrt{1 + a_{n+1}} \leq \sqrt{1 + a_n} = a_{n+1}.$$

By induction this shows $\{a_n\}_{n=1}^{\infty}$ is increasing.

Second, the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded (from above). We show by induction that $a_n \leq 2, \forall n \geq 1$. This is obviously true for $a_1 = 1$. Suppose $a_n \leq 2$ holds. Then by the recurrence relation we have

$$a_{n+1} = \sqrt{1 + a_n} \leq \sqrt{1 + 2} \leq 2.$$

By induction this shows $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2.

Now by the monotone convergence theorem (Theorem 2.4.4), $\{a_n\}_{n=1}^{\infty}$ converges to a number, say A. Taking the limits (as $n \to \infty$) of both sides of the recursions relation

$$a_{n+1} = \sqrt{1 + a_n},$$

we get

$$A = \sqrt{1 + A}.$$

Solving this yields $A = \frac{1 \pm \sqrt{5}}{2}$. Since $a_n \geq 1$, $\forall n \geq 1$, we have $A \geq 1$. Therefore $A \neq \frac{1-\sqrt{5}}{2} < 0$. So we must have

$$A = \frac{1 + \sqrt{5}}{2}.$$

Summarizing, we have shown that

$$\lim_{n \to \infty} a_n = \frac{1 + \sqrt{5}}{2}.$$

2.5#2 The fact that $A \geq 0$ follows from Theorem 2.2.1(f) with a_n taken to be constantly 0, and b_n taken to be the a_n in this problem. Now we show by definition that

$$\lim_{n \to \infty} \sqrt{a_n} = \sqrt{A}.$$

Given $\varepsilon > 0$, we need to find n^* such that

$$|\sqrt{a_n} - \sqrt{A}| < \varepsilon, \ \forall n \geq n^*.$$

Consider two different cases.

Case 1: $A = 0$. In this case we can find n^* such that

$$|a_n - 0| = a_n < \varepsilon^2, \ \forall n \geq n^*$$

by the convergence of $\{a_n\}_{n=1}^{\infty}$. Taking the square roots of both sides, we get

$$\sqrt{a_n} < \varepsilon, \ \forall n \geq n^*,$$

that is,

$$|\sqrt{a_n} - \sqrt{A}| < \varepsilon, \ \forall n \geq n^*,$$

as desired.

Case 2: $A > 0$. In this case we can write

$$\sqrt{a_n} - \sqrt{A} = (\sqrt{a_n} - \sqrt{A})\frac{\sqrt{a_n} + \sqrt{A}}{\sqrt{a_n} + \sqrt{A}} = \frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}.$$

So

$$|\sqrt{a_n} - \sqrt{A}| = \frac{|a_n - A|}{\sqrt{a_n} + \sqrt{A}} \leq \frac{|a_n - A|}{\sqrt{A}}.$$

By the convergence of $\{a_n\}_{n=1}^{\infty}$, there exists n^* such that

$$|a_n - A| < \varepsilon\sqrt{A}, \ \forall n \geq n^*.$$

Consequently, we have

$$|\sqrt{a_n} - \sqrt{A}| \leq \frac{|a_n - A|}{\sqrt{A}} < \varepsilon, \ \forall n \geq n^*,$$

as desired.

2.5#3 Given $\varepsilon > 0$, we need to find n^* such that

$$b_n = \sup\{|a_m - a_n| : m \geq n\} < \varepsilon, \ \forall n \geq n^*.$$

Since $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. By definition, there exists n^* such that

$$|a_n - a_m| < \varepsilon/2, \ \forall m \geq n \geq n^*.$$

Therefore for any fixed $n \geq n^*$, $\varepsilon/2$ is an upper bound for the set

$$\{|a_m - a_n| : m \geq n\}.$$

It follows that the least upper bound

$$\sup\{|a_m - a_n| : m \geq n\} \leq \varepsilon/2.$$

This shows

$$b_n \leq \varepsilon/2 < \varepsilon, \ \forall n \geq n^*.$$

By definition, $\{b_n\}_{n=1}^{\infty}$ converges to 0.