2.6#1 (a) $\{b_n\}$ is a subsequence of $\{a_n\}$, since $b_n = a_{2n+1}$.
(b) $\{b_n\}$ is not a subsequence of $\{a_n\}$, since $b_2 = 1/\sqrt{2}$ is not equal to any of the a_n's.
(c) $\{b_n\}$ is not a subsequence of $\{a_n\}$, since $b_1 = 1/3$ is not equal to any of the a_n's.

2.8#43 Assume for a contradiction that $\{a_n + b_n\}$ converges, say

$$\lim_{n\to\infty}(a_n + b_n) = C$$

for some number C. We also know that $\{a_n\}$ converges, say

$$\lim_{n\to\infty} a_n = A.$$

By Theorem 2.2.1(a), we then have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\{(a_n + b_n) - a_n\} = C - A.$$

This contradicts the assumption that $\{b_n\}$ diverges. Thus $\{a_n + b_n\}$ must diverge.

3.1#5(a) We claim that

$$\lim_{x\to\infty}\frac{-x^2}{2x^2-3} = \frac{-1}{2}.$$

By definition, for any $\varepsilon > 0$, we need to find $M > 0$ such that

$$\left|\frac{-x^2}{2x^2-3} - \frac{-1}{2}\right| < \varepsilon,\ \forall x \ge M.$$

However, we have

$$\frac{-x^2}{2x^2-3} - \frac{-1}{2} = \frac{-x^2}{2x^2-3} + \frac{1}{2} = \frac{-3}{2(2x^2-3)}.$$

So if $2x^2 - 3 > 0$, or $x > \sqrt{3/2}$, we have

$$\left|\frac{-x^2}{2x^2-3} - \frac{-1}{2}\right| = \frac{3}{2(2x^2-3)}.$$

If $x \ge M > \sqrt{3/2}$, then we can bound this by

$$\frac{3}{2(2x^2-3)} \le \frac{3}{2(2M^2-3)}.$$

If we set

$$M = \sqrt{\frac{\frac{3}{2\varepsilon}+3}{2}} > \sqrt{3/2},$$

then

$$\frac{3}{2(2M^2-3)} = \varepsilon.$$

Summarizing, for all $x \geq M$, we obtain that

$$\left| \frac{-x^2}{2x^2-3} - \frac{-1}{2} \right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof of the claim.

3.1#5(f) We claim that

$$\lim_{x \to \infty} \cos(x)$$

does not exist. Indeed, by Theorem 3.1.6, if the limit exists, say equals L, then we would have

$$\lim_{n \to \infty} \cos(x_n) = L$$

for any sequence $\{x_n\}$ with

$$\lim_{n \to \infty} x_n = \infty.$$

However, if $x_n = 2\pi n$, we get

$$\lim_{n \to \infty} \cos(x_n) = \lim_{n \to \infty} \cos(2\pi n) = 1.$$

On the other hand, if $x_n = 2\pi n + \pi$, we get

$$\lim_{n \to \infty} \cos(x_n) = \lim_{n \to \infty} \cos(2\pi n + \pi) = -1.$$

Thus $L = 1 = -1$, which is a impossible.