3.1#12 Given $\varepsilon > 0$, we need to find $M > a$ such that

$$|f(g(x)) - L| < \varepsilon, \ \forall x \geq M.$$

By the assumption that $\lim_{y\to\infty} f(y) = L$, there exists $K > a$ such that

$$|f(y) - L| < \varepsilon, \ \forall y \geq K.$$

Therefore if $y = g(x) \geq K$, we would have

$$|f(g(x)) - L| < \varepsilon.$$

On the other hand, by the assumption that $\lim_{x\to\infty} g(x) = \infty$, for this K there exists $M > a$ such that

$$g(x) \geq K, \ \forall x \geq M.$$

So, combining the above, we see that

$$|f(g(x)) - L| < \varepsilon$$

as long as $x \geq M$. This completes the proof.

3.2#1(d) We prove that

$$\lim_{x\to 0} \frac{x^2}{|x|} = 0.$$

By definition, given $\varepsilon > 0$, we need to find $\delta > 0$ such that

$$\left| \frac{x^2}{|x|} - 0 \right| < \varepsilon, \ \forall x \text{ with } 0 < |x - 0| < \delta,$$

that is,

$$\frac{x^2}{|x|} < \varepsilon, \ \forall x \text{ with } 0 < |x| < \delta.$$

Notice that

$$\frac{x^2}{|x|} = \frac{|x|^2}{|x|} = |x|.$$

So, choosing $\delta = \varepsilon$ gives that

$$\frac{x^2}{|x|} = |x| < \varepsilon, \quad \text{whenever } 0 < |x| < \delta.$$

This completes the proof.

3.2#5 (a) The example

$$1 = \lim_{x\to 0} |x| \cdot \frac{1}{|x|} \stackrel{?}{=} \left(\lim_{x\to 0} |x|\right) \left(\lim_{x\to 0} \frac{1}{|x|}\right) = 0 \cdot (\text{anything})$$

shows that "$0 \cdot$ (anything)" is not necessarily 0.
(b) We show that

$$\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0.$$

Notice that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1, \ \forall x \neq 0.$$

Multiplying both sides by $|x| > 0$, we get

$$-|x| \leq x \cdot \sin\left(\frac{1}{x}\right) \leq |x|, \ \forall x \neq 0.$$

Since

$$\lim_{x \to 0}(-|x|) = 0 = \lim_{x \to 0} |x|,$$

by the squeeze theorem, we can conclude

$$\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0,$$

as desired.