#1 (a) By definition, f is *continuous at* $x = 0$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(0)| < \varepsilon \text{ whenever } |x - 0| = |x| < \delta.$$

(b) Applying the definition in (a) with $\varepsilon = \frac{1}{2}$, we find $\delta > 0$ such that

$$|f(x) - 1| < \frac{1}{2} \text{ whenever } |x| < \delta.$$

Since

$$|f(x) - 1| < \frac{1}{2} \quad \Longleftrightarrow \quad -\frac{1}{2} < f(x) - 1 < \frac{1}{2},$$

we get, in particular,

$$f(x) > \frac{1}{2}.$$

This shows

$$f(x) > \frac{1}{2} \text{ whenever } |x| < \delta,$$

as desired.

#2 (a) Suppose to the contrary that there exists $x \in [0, 1]$ such that $f(x) \leq 0$. If $f(x) < 0$, then we must have $x > 0$ since it is assumed that $f(0) > 0$. Applying the Intermediate Value Theorem to the interval $[0, x]$, we see that, since f is continuous on $[0, x]$ and has different signs at the endpoints, there must exist $c \in (0, x)$ such that $f(c) = 0$. But this contradicts the assumption that f never equals 0 on $[0, 1]$. If $f(x) = 0$, then we get a contradiction as well. Therefore we must have $f(x) > 0$ for all $x \in [0, 1]$.
(b) By the Extreme Value Theorem, since f is continuous on $[0, 1]$, it attains its minimum on $[0, 1]$, i.e. there exists $c \in [0, 1]$ such that

$$\min_{x \in [0,1]} f(x) = f(c)$$

On the other hand, by (a), for this c we have

$$f(c) > 0.$$

So if we let

$$\varepsilon = \min_{x \in [0,1]} f(x),$$

then it holds $\varepsilon > 0$ and

$$f(x) \geq \varepsilon \text{ for all } x \in [0, 1].$$

This proves (b).

#3 (a) By definition, f is *uniformly continuous on* $[0, 1]$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } x, y \in [0, 1] \text{ and } |x - y| < \delta.$$

(b) Suppose f is $\frac{1}{2}$-Hölder continuous on $[0,1]$. Then for any given $\varepsilon > 0$, we have

$$|f(x) - f(y)| \leq C|x-y|^{1/2} < \varepsilon$$

provided that $x, y \in [0,1]$ and

$$C|x-y|^{1/2} < \varepsilon,$$

or equivalently,

$$|x-y| < \left(\frac{\varepsilon}{C}\right)^2.$$

Therefore, choosing

$$\delta = \left(\frac{\varepsilon}{C}\right)^2 > 0,$$

we have

$$|f(x) - f(y)| < \varepsilon \text{ whenever } x, y \in [0,1] \text{ and } |x-y| < \delta.$$

This shows f is uniformly continuous on $[0,1]$.

#4 (a) By definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}.$$

(b) By (a), since $f(0) = 0$, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x^2}\right)}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right).$$

Since

$$-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1, \ \forall x \neq 0,$$

we have

$$-|x| \leq x \sin\left(\frac{1}{x^2}\right) \leq |x|, \ \forall x \neq 0.$$

Therefore, by the Squeeze Theorem,

$$f'(0) = \lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) = 0.$$

#5 See Theorem 5.2.1(b), in which may take D to be any open interval containing $x = a$.

#6 (a) See Theorem 5.3.3.
(b) Let $x, y \in (a,b)$, $x \neq y$. We are going to show that $f(x) \neq f(y)$. Without loss of generality, we may assume $x < y$. Since f is differentiable on (a,b), it follows that f is continuous on $[x,y]$ and differentiable on (x,y). By the Mean Value Theorem, there exists $c \in (x,y)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(c),$$

or equivalently,

$$f(x) - f(y) = f'(c)(x - y).$$

However, by the assumption we have $f'(c) \neq 0$. Therefore, $f(x) - f(y) \neq 0$, and so $f(x) \neq f(y)$. This proves (b).

#7 (a) Since f is a polynomial, the function $F(x)$ is differentiable on $(-\infty, \infty)$. In particular, it is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Moreover, we have

$$F(0) = f(0) + f'(0) + f(1), \quad F(1) = f(1).$$

By assumption, $f(0) = 0$, $f'(0) = 0$. So we have

$$F(0) = F(1) = f(1).$$

It now follows from Rolle's theorem that there exists $c \in (0, 1)$ such that

$$F'(c) = 0.$$

(b) By direct computation, we have

$$F'(x) = f''(x)(1 - x) - 2f(1)(1 - x).$$

Therefore, with $c \in (0, 1)$ as in (a), we have

$$F'(c) = f''(c)(1 - c) - 2f(1)(1 - c) = 0.$$

However, this implies

$$f''(c)(1 - c) = 2f(1)(1 - c),$$

or, since $1 - c \neq 0$,

$$f''(c) = 2f(1).$$

From this we obtain

$$f(1) = \frac{f''(c)}{2},$$

as desired.

#8 (a) By inspection, we have (why?)

$$U(P_n, f) = \begin{cases} \frac{n+2}{2n} & \text{if } n \text{ is even,} \\ \frac{n+1}{2n} & \text{if } n \text{ is odd,} \end{cases}$$

and

$$L(P_n, f) = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2n} & \text{if } n \text{ is odd.} \end{cases}$$

(b) Notice that for $n = 1, 2, \cdots$, we have

$$L(P_n, f) \leq \underline{\int_0^1} f \leq \overline{\int_0^1} f \leq U(P_n, f).$$

On the other hand, it follows from (a) that (why?)

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = \frac{1}{2}.$$

So, by the Squeeze Theorem, it must hold that

$$\underline{\int_0^1} f = \overline{\int_0^1} f = \frac{1}{2}.$$

This shows $f \in R[a, b]$ and moreover

$$\int_0^1 f = \frac{1}{2}.$$

#9 See Theorem 6.2.2.

#10 (a) We will show that

$$\overline{\int_a^b} f \leq \overline{\int_a^b} g.$$

Since g is assumed to be Riemann integrable, it will then follow that

$$\overline{\int_a^b} f \leq \int_a^b g \left(= \overline{\int_a^b} g \right).$$

To show the first inequality, let $P = \{x_k\}_{k=1}^n$ be any given partition of the interval $[a, b]$. By assumption, $f(x) \leq g(x)$, $\forall x \in [a, b]$. So, for $k = 1, \cdots, n$, we have

$$M_k(f) = \sup_{[x_{k-1}, x_k]} f \leq \sup_{[x_{k-1}, x_k]} g = M_k(g),$$

and therefore

$$U(P, f) = \sum_{k=1}^{n} M_k(f)\Delta x_k \leq \sum_{k=1}^{n} M_k(g)\Delta x_k = U(P, g).$$

From this we obtain, since $\overline{\int_a^b} f \leq U(P, f)$,

$$\overline{\int_a^b} f \leq U(P, g).$$

Since P is arbitrary and $\overline{\int_a^b} g = \inf_P U(P, g)$, this implies

$$\overline{\int_a^b} f \leq \overline{\int_a^b} g,$$

the claimed inequality.
(b) If $f(x) \geq 0, \ \forall x \in [a, b]$. Then by Exercise 6.1.6,

$$\underline{\int_a^b} f \geq 0.$$

So, if $\int_a^b g = 0$,

$$0 \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \int_a^b g = 0,$$

where we have used (a) in the last inequality. However, this implies

$$\underline{\int_a^b} f = \overline{\int_a^b} f = 0.$$

Therefore, under the assumptions, we must have $f \in R[a, b]$ and

$$\int_a^b f = 0.$$

This proves (b).

#11 See Theorem 6.4.2. Understand the precise meaning of the assumptions on f and f'.