5.3#6(b) ($\Rightarrow$) Suppose $f'(c) > 0$ for all $c \in (a, b)$. Then for any $x, y \in (a, b)$ with $x < y$, by the Mean Value Theorem we have, for some $c \in (x, y)$,

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

or, equivalently,

$$f(y) - f(x) = f'(c)(y - x).$$

Since $f'(c) > 0$ and $y - x > 0$, it follows that $f(y) - f(x) > 0$, that is, $f(y) > f(x)$. This shows f is strictly increasing, since $f(y) > f(x)$ whenever $x > y$.
($\not\Leftarrow$) The converse statement is false. A counter-example is given by $f(x) = x^3$ which is strictly increasing on $(-1, 1)$, but with $f'(0) = 0$.

5.3#17(e) Take $f(x) = \sin(x)$. Note that f is differentiable on $(-\infty, \infty)$ and $f'(x) = \cos(x)$. Suppose $x < y$. By the Mean Value Theorem, we have for some $c \in (x, y)$,

$$f(y) - f(x) = f'(c)(x - y).$$

Therefore

$$|\sin(y) - \sin(x)| = |\cos(c)||x - y|.$$

But $|\cos(c)| \leq 1$. So we obtain

$$|\sin(y) - \sin(x)| \leq |x - y|.$$

If $x > y$, then reverting x and y in the argument above gives that same bound. Finally, if $x = y$, then trivially

$$|\sin(y) - \sin(x)| = 0 \leq |x - y|.$$

Therefore $|\sin(y) - \sin(x)| \leq |x - y|$ holds for all x and y.

For the second part of the question, notice that since $\sin(x)$ is an odd function, we have $\sin(x) = -\sin(-x)$. Therefore, applying the inequality proven above, we get

$$|\sin(y) + \sin(x)| = |\sin(y) - \sin(-x)| \leq |y - (-x)| = |y + x|.$$

This shows

$$|\sin(y) + \sin(x)| \leq |y + x|.$$

5.4#9 Suppose $f''(x) = 0$. Then by Corollary 5.3.7(a), we have

$$f'(x) = a$$

for some constant a. Let $g(x) = ax$, then $f'(x) = g'(x)$ for all x. So by Corollary 5.3.7(b), we must have

$$f(x) = g(x) + b$$

for some constant b. This shows

$$f(x) = ax + b,$$

as desired.