2.1#2(d) We show that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to $1/2$. By definition, given any $\varepsilon > 0$, we need to find an n^* such that

$$\left|\frac{n}{2n+\sqrt{n}} - \frac{1}{2}\right| < \varepsilon, \ \forall n \geq n^*.$$

Since

$$\left|\frac{n}{2n+\sqrt{n}} - \frac{1}{2}\right| = \left|\frac{-\sqrt{n}}{2(2n+\sqrt{n})}\right| = \frac{1}{4\sqrt{n}+2},$$

the condition above is equivalent to

$$\frac{1}{4\sqrt{n}+2} < \varepsilon, \ \forall n \geq n^*.$$

Since

$$\frac{1}{4\sqrt{n}+2} < \frac{1}{4\sqrt{n}},$$

it suffices to have

$$\frac{1}{4\sqrt{n}} < \varepsilon, \ \forall n \geq n^*,$$

or equivalently,

$$\frac{1}{(4\varepsilon)^2} < n^*.$$

Thus choosing for instance

$$n^* = \left\lceil \frac{1}{(4\varepsilon)^2} \right\rceil + 1$$

would verify the definition of convergence. This completes the proof.

2.1#2(g) We show that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0. By definition, given any $\varepsilon > 0$, we need to find an n^* such that

$$\left|(\sqrt{n+1} - \sqrt{n}) - 0\right| < \varepsilon, \ \forall n \geq n^*.$$

Since

$$\left|(\sqrt{n+1} - \sqrt{n}) - 0\right| = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}},$$

the condition above is equivalent to

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} < \varepsilon, \ \forall n \geq n^*.$$

Since

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}},$$

it suffices to have

$$\frac{1}{\sqrt{n}} < \varepsilon, \ \forall n \geq n^*,$$

or equivalently,

$$\frac{1}{\varepsilon^2} < n^*.$$

Thus choosing for instance

$$n^* = \left\lceil \frac{1}{\varepsilon^2} \right\rceil + 1$$

would verify the definition of convergence. This completes the proof.

2.1#4 By definition, the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0 if and only if for any given $\varepsilon > 0$, there exists an n^* such that

$$|a_n - 0| < \varepsilon, \ \forall n \geq n^*.$$

On the other hand, the sequence $\{|a_n|\}_{n=1}^{\infty}$ converges to 0 if and only if for any given $\varepsilon > 0$, there exists an n^* such that

$$||a_n| - 0| < \varepsilon, \ \forall n \geq n^*.$$

Since

$$|a_n - 0| = |a_n| = ||a_n| - 0|\,,$$

the two definitions actually coincide. Therefore the convergences of $\{a_n\}$ and $\{|a_n|\}$ to 0 are equivalent. This completes the proof.

2.1#6 By definition, the sequence $\{a_n\}_{n=1}^{\infty}$ converges to A if and only if for any given $\varepsilon > 0$, there exists an n^* such that

$$|a_n - A| < \varepsilon, \ \forall n \geq n^*.$$

On the other hand, the sequence $\{a_n - A\}_{n=1}^{\infty}$ converges to 0 if and only if for any given $\varepsilon > 0$, there exists an n^* such that

$$|(a_n - A) - 0| < \varepsilon, \ \forall n \geq n^*.$$

Since

$$|(a_n - A) - 0| = |a_n - A|\,,$$

the two definitions actually coincide, and therefore the equivalence of the convergences. This completes the proof.