2.1#5 Assume that $\{a_n\}_{n=1}^{\infty}$ converges to A. By definition for any $\varepsilon > 0$, there exists n^* such that

$$|a_n - A| < \varepsilon, \ \forall n \geq n^*.$$

By the triangle inequality we have

$$||a_n| - |A|| \leq |a_n - A|.$$

So it follows that

$$||a_n| - |A|| < \varepsilon, \ \forall n \geq n^*.$$

This shows that $\{|a_n|\}_{n=1}^{\infty}$ converges to $|A|$.

The converse is false. Consider $a_n = (-1)^n$. Then $\{|a_n| = 1\}_{n=1}^{\infty}$ converges to 1. But $\{a_n = (-1)^n\}_{n=1}^{\infty}$ diverges.

2.1#12 By Theorem 2.1.12, since $\{a_n\}_{n=1}^{\infty}$ converges to $A \neq 0$, there exists n^* such that

$$|a_n| \geq \frac{|A|}{2}, \ \forall n \geq n^*.$$

From this we get

$$\frac{1}{|a_n|} \leq \frac{2}{|A|}, \ \forall n \geq n^*.$$

On the other hand, by the assumption we have $a_n \neq 0, \ \forall n \geq 1$. So we can take

$$M = \max\left\{ \frac{1}{|a_1|}, \cdots, \frac{1}{|a_{n^*-1}|}, \frac{2}{|A|} \right\},$$

so that

$$\frac{1}{|a_n|} \leq M, \ \forall n \geq 1.$$

This shows $\{\frac{1}{a_n}\}_{n=1}^{\infty}$ is a bounded sequence.

2.7#43 Assume for a contradiction that $\{a_n + b_n\}$ converges, say

$$\lim_{n\to\infty} (a_n + b_n) = C$$

for some number C. We also know that $\{a_n\}$ converges, say

$$\lim_{n\to\infty} a_n = A.$$

By Theorem 2.2.1(a), we then have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \{(a_n + b_n) - a_n\} = C - A.$$

This contradicts the assumption that $\{b_n\}$ diverges. Thus $\{a_n + b_n\}$ must diverge.