2.2#5 Since $\{b_n\}_{n=1}^{\infty}$ is bounded, there exists $M > 0$ such that

$$|b_n| \le M, \ \forall n \ge 1.$$

It follows that

$$0 \le |a_n b_n| \le M|a_n|, \ \forall n \ge 1.$$

Since $\{a_n\}_{n=1}^{\infty}$ converges to 0, by Theorem 2.1.14 we have $\{|a_n|\}_{n=1}^{\infty}$ converges 0, and therefore

$$\lim_{n\to\infty} M|a_n| = 0.$$

By the squeeze theorem, it follows that

$$\lim_{n\to\infty} |a_n b_n| = 0.$$

By Theorem 2.1.14 again, we then have

$$\lim_{n\to\infty} a_n b_n = 0,$$

as desired.

2.2#8(b) Consider

$$a_n = 0, \quad b_n = \frac{1}{n}, \quad n \ge 1.$$

Then

$$a_n < b_n, \ \forall n \ge 1.$$

But

$$A = \lim_{n\to\infty} a_n = 0, \quad B = \lim_{n\to\infty} b_n = 0.$$

Thus it is incorrect to conclude $A < B$.

2.2#12 Consider

$$a_n = \frac{1}{n}, \quad b_n = n, \quad n \ge 1.$$

Then, clearly

$$\lim_{n\to\infty} a_n = 0.$$

On the other hand,

$$a_n b_n = 1, \ \forall n \ge 1.$$

Thus it is *not* necessarily true that

$$\lim_{n\to\infty} a_n b_n = 0.$$

2.3#2(a) To show

$$\lim_{n\to\infty} (a_n + b_n) = \infty,$$

for any given $M > 0$, we need to find an n^* such that

$$a_n + b_n > M, \ \forall n \geq n^*.$$

However, we know that $\{a_n\}$ diverges to ∞. Therefore, with K denoting a lower bound of $\{b_n\}$, we can find an n^* such that

$$a_n > M - K, \ \forall n \geq n^*.$$

It follows that, with this n^*, for all $n \geq n^*$ we have

$$a_n + b_n > (M - K) + K = M, \ \forall n \geq n^*,$$

as desired.

The limit

$$\lim_{n \to \infty} (a_n + b_n) = \infty$$

may *not* hold if $\{b_n\}$ is not assumed bounded below. For instance, consider

$$a_n = n, \quad b_n = -n.$$

Then clearly

$$\lim_{n \to \infty} (a_n + b_n) = 0 \neq \infty.$$