**2.4#1** An example is given by

$$a_n = n + (-1)^n, \ n \geq 1$$

which diverges to $+\infty$ by Theorem 2.3.3(a), but is not eventually increasing as

$$a_{2n} - a_{2n+1} = 1 > 0.$$

**2.4#2** An example is given by

$$a_n = 1 - \frac{1}{n}, \ n \geq 1$$

which converges to 1, but does not attain a maximum since it is strictly increasing.

**2.5#5** ($\Rightarrow$) Let $M = \sup S$. Suppose $M < \infty$. Clearly, $M$ is an upper bound of $S$. To find $\{a_n\}$, notice that for any $n \geq 1$, $M - \frac{1}{n}$ is not an upper bound of $S$ by the minimality of $M$. Therefore, there must exist a number in $S$, say $a_n$, such that

$$M - \frac{1}{n} < a_n.$$

On the other hand, since $M$ is an upper bound of $S$ and $a_n \in S$, we also have $a_n \leq M$. Thus

$$M - \frac{1}{n} < a_n \leq M, \ \forall n \geq 1.$$

Taking $n \to \infty$, by the squeeze theorem, it follows that

$$\lim_{n\to\infty} a_n = M.$$

The case $M = \infty$ can be treated similarly.
($\Leftarrow$) To show that $M$ is the least upper bound, it suffices to show any other upper bound $M'$ satisfies

$$M \leq M'.$$

Since $M'$ is an upper bound of $S$ and $a_n \in S$, we have, for all $n \geq 1$,

$$a_n \leq M'.$$

Taking $n \to \infty$, by Theorem 2.2.1(f), we obtain,

$$\lim_{n\to\infty} a_n \leq M'.$$

By assumption $\lim_{n\to\infty} a_n = M$. Thus

$$M \leq M',$$

as desired.