2.6#5 Suppose the sequence $\{a_n\}_{n=1}^{\infty}$ is unbounded above. We want to find a subsequence $\{a_{n_k}\}$ which increases to ∞. It suffices to describe how to find the indexes $\{n_k\}$. We will do this inductively in k.

First, we let

$$n_1 = 1.$$

Assuming n_k have been chosen, we now choose n_{k+1}. Let

$$M = 1 + \max\{k, a_1, a_2, \cdots, a_{n_k}\} > 0.$$

Since M cannot be an upper bound of $\{a_n\}_{n=1}^{\infty}$, there must exist an n such that

$$a_n \geq M.$$

Let n_{k+1} be this n. Note that, since

$$M > a_1, a_2, \cdots, a_{n_k},$$

we must have $n_{k+1} > n_k$. This finishes our choice of n_{k+1}.

By induction, this generates a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$. Note that this is indeed a subsequence as n_k is strictly increasing in k. On the other hand, since (by our choice of M)

$$a_{n_k} \geq k, \ \forall k \geq 2,$$

and

$$a_{n_{k+1}} > a_{n_k}, \ k \geq 1,$$

we also have, monotonically,

$$\lim_{k\to\infty} a_{n_k} = \infty.$$

Therefore $\{a_{n_k}\}_{k=1}^{\infty}$ satisfies the desired properties, and the proof is now complete.

2.7#17 Suppose

$$\lim_{n\to\infty} a_n = A \neq 0.$$

We want to show that $\{b_n\} = \{(-1)^n a_n\}$ diverges. Assume for a contradiction that $\{(-1)^n a_n\}$ converges to L. Then along the subsequences $\{2k\}$ and $\{2k+1\}$ we have, respectively,

$$\lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} (-1)^{2k} a_{2k} = L,$$

$$\lim_{k\to\infty} b_{2k+1} = \lim_{k\to\infty} (-1)^{2k+1} a_{2k+1} = L.$$

Since $(-1)^{2k} = 1$ and $(-1)^{2k+1} = -1$, it follows that

$$\lim_{k\to\infty} a_{2k} = L, \quad \lim_{k\to\infty} (-a_{2k+1}) = L.$$

However, the left-hand sides equal A and $-A$ respectively. So we obtain

$$A = -A.$$

This contradicts the assumption $A \neq 0$. Thus $\{(-1)^n a_n\}$ cannot converge, i.e. it diverges.