**4.1#6** (d) Consider the Heaviside function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0. \end{cases}$$

Then $f$ and $g$ are both discontinuous at $x = 0$. However,

$$f(x)g(x) = 0, \ \forall x \in (-\infty, \infty).$$

In particular $fg$ is continuous on $(-\infty, \infty)$.
(e) Let $f$ and $g$ be as above. Then

$$f(x) + g(x) = 1, \ \forall x \in (-\infty, \infty).$$

Thus $f + g$ is continuous on $(-\infty, \infty)$, even though $f$ and $g$ are not.

**4.1#8(a)** (d) Since $f$ is continuous at $x = a$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta, \ x \in D.$$

Note that this implies

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon.$$

Since $f(a) > 0$, we can choose $\varepsilon = \frac{f(a)}{2} > 0$ above and obtain a $\delta > 0$ such that

$$f(a) - \frac{f(a)}{2} < f(x) < f(a) + \frac{f(a)}{2} \text{ whenever } |x - a| < \delta, \ x \in D.$$

With this $\delta > 0$, we then have

$$\frac{f(a)}{2} < f(x) \text{ whenever } |x - a| < \delta, \ x \in D,$$

which is the desired conclusion.

**4.3#1(a)**

(i) $f(x) = 1/x$ is continuous on the bounded, open interval $(0, 1)$. But $f$ is unbounded.
(ii) $f(x) = x$ is continuous on the unbounded, closed interval $[0, \infty)$. But $f$ is unbounded.
(iii) Define $f(x) = 1/x$ if $0 < x \leq 1$ and $f(x) = 0$ if $x = 0$. Then $f(x)$ is a discontinuous function on the closed interval $[0, 1]$, meanwhile $f$ is unbounded.