4.3#8 (Visualize the proof on your scratch paper) Suppose f is continuous and one-to-one on $[a, b]$. We show that f must be strictly monotone. Without loss of generality, we assume that $f(a) < f(b)$. (The case $f(a) > f(b)$ can be treated similarly.)

First, observe that, for any $c \in (a, b)$ it must hold that $f(a) < f(c) < f(b)$. This is because otherwise, if $f(c) < f(a)(< f(b))$, we would then have, by the Intermediate Value Theorem, that there exists $d \in (c, b)$ such that $f(d) = f(a)$, contradicting the injectivity of f. If instead $f(c) > f(b)(> f(a))$, then we deduce a contradiction in a similar way. Note that $f(c)$ cannot equal $f(a)$ or $f(b)$ since f is one-to-one.

Now we can show that $f(x) < f(y)$ whenever $a < x < y < b$, thus proving that f is strictly increasing. Assume otherwise we have $f(x) > f(y)(> f(a))$. Then, by the Intermediate Value Theorem again, there exists $c \in (a, x)$ such that $f(c) = f(y)$, which contradicts the injectivity of f. This completes the proof.

4.4#1(a) $f(x) = 1/x$ is not uniformly continuous on $(0, 1]$. This is because, otherwise, by Corollary 4.4.8, f can be extended to be a continuous function on the closed interval $[0, 1]$, which would then imply that f is bounded. However, it is easy to see that $f(x) = 1/x$ is not bounded on $(0, 1]$.

4.3#11 Since f is continuous and nonvanishing on $[a, b]$, by **Exercise 4.1#5**, so is $|f|$. On the other hand, by Theorem 4.1.8(c), the function $1/|f|$ is also continuous on $[a, b]$. Thus, by Theorem 4.3.4, $1/|f|$ must be bounded on $[a, b]$, i.e. we have

$$\frac{1}{|f(x)|} \leq M, \ \forall x \in [a, b]$$

for some positive number M. Consequently,

$$|f(x)| \geq \frac{1}{M}, \ \forall x \in [a, b].$$

This shows that f is bounded away from zero on $[a, b]$.

4.4#8(a) Suppose f is a Lipschitz function on I, i.e. there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \ \forall x, y \in I.$$

Then for any given $\varepsilon > 0$, we can let

$$\delta = \frac{\varepsilon}{L}$$

so that, whenever $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq L|x - y| < L\delta = L \cdot \frac{\varepsilon}{L} = \varepsilon.$$

By definition, this shows f is uniformly continuous on I.

(b) Consider $f(x) = \sqrt{x}$ on $[0, 1]$. Then f is continuous on $[0, 1]$, therefore is uniformly continuous on $[0, 1]$ by Theorem 4.4.6. However, f is not Lipschitz on $[0, 1]$. Indeed, if f were Lipschitz, i.e. there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \ \forall x, y \in [0, 1].$$

Then taking $x = 0$ and $y \in (0, 1]$, we get

$$\sqrt{y} \leq Ly,$$

or

$$1 \leq L\sqrt{y}.$$

This is not possible since the right-hand side tends to 0 as $y \to 0^+$.