1. Find the limit. Use L'Hospital's Rule where appropriate.

(a) $\lim_{x\to\infty} xe^{-\sqrt{x}} = \boxed{0}$

Solution: write $xe^{-\sqrt{x}} = \frac{x}{e^{\sqrt{x}}}$ and apply L'Hospital's Rule twice.

(b) $\lim_{x\to\infty} x^{1/\sqrt{x}} = \boxed{1}$

Solution: write $x^{1/\sqrt{x}} = e^{\frac{\ln x}{\sqrt{x}}}$ and find the limit of the exponent (which equals 0).

(c) $\lim_{x\to 0^+} (1+x)^{1/x^2} = \boxed{\infty}$

Solution: write $(1+x)^{1/x^2} = e^{\frac{\ln(1+x)}{x^2}}$ and find the limit of the exponent.

2. Evaluate the integral using integration by parts.

(a) $\int t^2 e^{-t} dt = \boxed{-(t^2+2t+2)e^{-t} + C}$

Solution: integrate by parts twice with $dv = e^{-t}dt$.

(b) $\int x^2 \sin(10x)dx = \boxed{\frac{1}{500}\big(-50x^2\cos(10x) + 10x\sin(10x) + \cos(10x)\big) + C}$

Solution: integrate by parts twice with $dv = \sin(10x)dx$.

(c) $\int e^{-x}\cos(10x)dx = \boxed{-\frac{e^{-x}}{101}\big(\cos(10x) - 10\sin(10x)\big) + C}$

Solution: integrate by parts twice and use the trick of combining identical terms.

3. Evaluate the trig integral.

(a) $\int \tan^2 x dx = \boxed{\tan(x) - x + C}$

Solution: write $\tan^2 x = \sec^2 x - 1$ and notice that $\int \sec^2 x dx = \tan x + C$.

(b) $\int \sin^3 x dx = \boxed{\frac{\cos^3 x}{3} - \cos x + C}$

Solution: write $\sin^3 x = (1-\cos^2 x)\sin x$ and apply the substitution $u = \cos x$.

(c) $\int \sec^4 x dx = \boxed{\frac{\tan^3 x}{3} + \tan x + C}$

Solution: write $\sec^4 x = (\tan^2 x + 1)\sec^2 x$ and apply the substitution $u = \tan x$.

4. Evaluate the integral using trig substitution.

(a) $\displaystyle\int \frac{1}{(1+x^2)^{3/2}}dx = \boxed{\frac{x}{\sqrt{1+x^2}} + C}$

Solution: letting $x = \tan\theta$ turns the integral into $\int \cos\theta d\theta = \sin\theta + C$. Now use a θ-triangle to show that $\sin\theta = \frac{x}{\sqrt{1+x^2}}$.

(b) $\displaystyle\int \sqrt{1-4x^2}dx = \boxed{\frac{x\sqrt{1-4x^2}}{2} + \frac{\arcsin(2x)}{4} + C}$

Solution: letting $2x = \sin\theta$ turns the integral into $\frac{1}{2}\int \cos^2\theta d\theta$. Using the half-angle formula this becomes $\frac{\sin(2\theta)}{8} + \frac{\theta}{4} + C$. Now use the double-angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$, together with $\sin\theta = 2x$, $\cos\theta = \sqrt{1-4x^2}$, to get the answer.

(c) $\displaystyle\int \frac{(4x^2-1)^{3/2}}{x}dx = \boxed{\frac{(4x^2-1)^{3/2}}{3} - \sqrt{4x^2-1} + \operatorname{arcsec}(2x) + C}$

Solution: letting $x = \frac{1}{2}\sec\theta$ turns the integral into $\int \tan^4\theta d\theta$. Writing $\tan^4\theta = (\sec^2\theta - 1)\tan^2\theta = \sec^2\theta\tan^2\theta - (\sec^2\theta - 1)$, this integral becomes $\frac{\tan^3\theta}{3} - (\tan\theta) + \theta + C$. Since $\tan\theta = \sqrt{4x^2-1}$, the answer follows.

5. Evaluate the integral using partial fractions.

(a) $\displaystyle\int \frac{x^3}{1+x^2}dx = \boxed{\frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C}$

Solution: long division gives $x^3 = x(1+x^2) - x$, and therefore $\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$. Integrating the last expression gives the answer.

(b) $\displaystyle\int \frac{x+1}{x^2-x}dx = \boxed{2\ln|x-1| - \ln|x| + C}$

Solution: use partial fractions to write $\frac{x+1}{x^2-x} = \frac{2}{x-1} - \frac{1}{x}$. Integrating this gives the answer.

(c) $\displaystyle\int \frac{x^2+1}{x^3-2x^2+x}dx = \boxed{-\frac{2}{x-1} + \ln|x| + C}$

Solution: use partial fractions to write $\frac{x^2+1}{x^3-2x^2+x} = \frac{2}{(x-1)^2} + \frac{1}{x}$. Integrating this gives the answer.

6. Evaluate the improper integral, if it is convergent.

(a) $\displaystyle\int_0^\infty \frac{1+x}{1+x^2}dx = \boxed{\infty}$

Solution: write the integral as $\int_0^\infty \frac{1}{1+x^2}dx + \int_0^\infty \frac{x}{1+x^2}dx$. The former equals $[\arctan x]_0^\infty = \frac{\pi}{2}$ and the latter equals $[\frac{1}{2}\ln(1+x^2)]_0^\infty = \infty$. The answer follows by summing the two.

(b) $\displaystyle\int_0^{1/e} \frac{1}{x(\ln x)^2}dx = \boxed{1}$

Solution: letting $u = \ln x$, the integral becomes $\int_{-\infty}^{-1} \frac{1}{u^2}du = [-\frac{1}{u}]_{-\infty}^{-1} = 1 - 0 = 1$.

(c) $\displaystyle\int_0^{\pi/2} \frac{1}{\cos x}dx = \boxed{\infty}$

Solution: $\int_0^{\pi/2} \frac{1}{\cos x}dx = \int_0^{\pi/2} \sec x dx = \big[\ln|\sec x + \tan x|\big]_0^{\pi/2} = \ln|\infty + \infty| - 0 = \infty$.

7. The region enclosed by the given curves is rotated about the x-axis. Find the volume of the resulting solid using cross-sections.

(a) $y = |x|,\ y = 2 - x^2$

Solution: the two curves intersect at $x = \pm 1$ and the curve $y = 2 - x^2$ provides the outer radii. So the volume equals $\int_{-1}^{1} \pi\big[(2-x^2)^2 - |x|^2\big]dx$. Now write $(2-x^2)^2 = 4 - 4x^2 + x^4$ and $|x|^2 = x^2$ to get the volume $= \boxed{\frac{76}{15}\pi}$.

(b) $y = \cos x,\ y = \sin x,\ 0 \le x \le \pi/4$

Solution: $y = \cos x$ provides the outer radii. So the volume equals $\int_0^{\pi/4} \pi\big[(\cos x)^2 - (\sin x)^2\big]dx = \pi\int_0^{\pi/4} \cos(2x)dx = \boxed{\frac{\pi}{2}}$.

8. Use the method of cylindrical shells to find the volume generated by rotating about the y-axis the region bounded by the given curves.

(a) $y = |x|,\ y = 2 - x^2$

Solution: the two curves intersect at $x = \pm 1$ and $y = 2 - x^2$ is the upper curve. By symmetry we can drop the portion of the region with negative x-component. So the volume equals $\int_0^1 2\pi x\big[(2-x^2) - x\big]dx = \boxed{\frac{5}{6}\pi}$.

(b) $y = \sqrt{x^2+1},\ y = 0,\ x = 0,\ x = 1$

Solution: the volume equals $\int_0^1 2\pi x\sqrt{x^2+1}dx = \boxed{\frac{2\pi}{3}(2^{3/2} - 1)}$.

9. Find the arc length of the curve.

(a) $\ln(\cos x),\ 0 \le x \le \pi/4$

Solution: since $[\ln(\cos x)]' = -\tan x$, the arc length equals $\int_0^{\pi/4} \sqrt{1+\tan^2 x}dx = \int_0^{\pi/4} \sec x dx = \big[\ln|\sec x + \tan x|\big]_0^{\pi/4} = \boxed{\ln(\sqrt{2}+1)}$.

(b) $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}, \ 1 \le x \le 2$

Solution: note that $1 + [dy/dx]^2 = 1 + (\frac{x}{2} - \frac{1}{2x})^2 = (\frac{x}{2} + \frac{1}{2x})^2$. Therefore the arc length equals $\int_1^2 \sqrt{1+[dy/dx]^2}dx = \int_1^2 (\frac{x}{2} + \frac{1}{2x})dx = [\frac{x^2}{4} + \frac{\ln x}{2}]_1^2 = \boxed{\dfrac{3}{4} + \dfrac{\ln 2}{2}}$.

10. Find the area of the surface obtained by rotating the curve about the specified axis.

(a) $y = \sqrt{1+e^x}, \ 0 \le x \le 1;$ about the x-axis

Solution: the surface area equals $\int_0^1 2\pi y ds = \int_0^1 2\pi\sqrt{1+e^x}\sqrt{1+(\frac{e^x}{2\sqrt{1+e^x}})^2}dx$. Combining the square roots the integrand becomes $2\pi\sqrt{(1+e^x)+\frac{e^{2x}}{4}} = \pi\sqrt{4+4e^x+e^{2x}} = \pi\sqrt{(2+e^x)^2} = \pi(2+e^x)$. So the area equals $\int_0^1 \pi(2+e^x)dx = \boxed{(1+e)\pi}$.

(b) $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}, \ 1 \le x \le 2;$ about the y-axis

Solution: the surface area equals $\int_1^2 2\pi x ds = \int_1^2 2\pi x\sqrt{1+[dy/dx]^2}dx$. By the computation in **9**(b), this equals $\int_1^2 2\pi x(\frac{x}{2} + \frac{1}{2x})dx = \boxed{\dfrac{10}{3}\pi}$.