Name: ______________________ Math 232 Quiz 5

1. (10 pts) Evaluate the integral.

(a) $\displaystyle\int \frac{x^3+1}{x^2+1}\,dx$

The fraction is not proper, so perform long division:

$$\begin{array}{r} x \\ \underbrace{x^2+1}_{\text{divisor}}\,\overline{)\,x^3+1} \\ x^3+x \\ \hline -x+1 \end{array}$$

$$\Rightarrow x^3+1 = \overbrace{x}^{\text{quotient}}(x^2+1) + \overbrace{(-x+1)}^{\text{remainder}}$$

$$\Rightarrow \frac{x^3+1}{x^2+1} = x + \frac{-x+1}{x^2+1}$$

$$\Rightarrow \int \frac{x^3+1}{x^2+1}\,dx = \int\Big(x + \underbrace{\frac{-x+1}{x^2+1}}_{\text{proper; irreducible}}\Big)dx$$

$$= \int\Big(x - \frac{x}{x^2+1} + \frac{1}{x^2+1}\Big)dx$$

$$= \boxed{\frac{x^2}{2} \underbrace{-\tfrac{1}{2}\ln(x^2+1)}_{(u=x^2+1)} + \arctan(x) + C}$$

(b) $\displaystyle\int \frac{3}{x^2-x-2}\,dx$

This is a proper fraction. $x^2-x+2 = (x-2)(x+1)$ (complete factorization)

Partial fraction: $\dfrac{3}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1}$ (distinct linear factors)

$$3 = A(x+1) + B(x-2) \quad \text{(multiply by } (x-2)(x+1))$$

$$\Rightarrow \begin{cases} @x=2: & 3 = 3A \\ @x=-1: & 3 = -3B \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$$

So, $\dfrac{3}{(x-2)(x+1)} = \dfrac{1}{x-2} - \dfrac{1}{x+1}$.

$$\int \frac{3}{x^2-x+2}\,dx = \int\Big(\frac{1}{x-2} - \frac{1}{x+1}\Big)dx = \boxed{\ln|x-2| - \ln|x+1| + C}$$

2. (10 pts) Find the limit. Use L'Hospital's Rule where appropriate.

(a) $\lim_{x \to 0} \frac{x - \ln(1+x)}{x^2}$ $\underset{L'H}{\overset{"\frac{0}{0}"}{=}}$ $\lim_{x \to 0} \frac{1 - \frac{1}{1+x}}{2x}$ $\underset{L'H}{\overset{"\frac{0}{0}"}{=}}$ $\lim_{x \to 0} \frac{\frac{1}{(1+x)^2}}{2}$ $= \boxed{\frac{1}{2}}$

(b) $\lim_{x \to \infty} x e^{-x}$ $= \lim_{x \to \infty} \frac{x}{e^x}$ $\underset{L'H}{\overset{"\frac{\infty}{\infty}"}{=}}$ $\lim_{x \to \infty} \frac{1}{e^x}$ $= \frac{1}{\infty} = \boxed{0}$

(c) $\lim_{x \to \infty} x^{\frac{1}{x}}$ $= \lim_{x \to \infty} e^{\frac{1}{x} \ln x}$ $= e^{\lim_{x \to \infty} \frac{\ln x}{x}}$ $\overset{L'H}{=} e^0 = \boxed{1}$

$$\left[\lim_{x \to \infty} \frac{\ln x}{x} \underset{L'H}{\overset{"\frac{\infty}{\infty}"}{=}} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0 \right]$$