Name: ____________________ Math 232 Quiz 8

1. Use the *(Limit) Comparison Test* to determine if the series converges or diverges.

(*a*) (3 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2+1}$

$$0 \leq \underbrace{\frac{|\sin n|}{n^2+1}}_{a_n} \leq \frac{1}{n^2+1} \leq \underbrace{\frac{1}{n^2}}_{b_n}$$

Since $\sum \frac{1}{n^2}$ converges,

by the comparison test,

$$\sum \frac{|\sin n|}{n^2+1}$$

converges too.

(*b*) (3 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Let $a_n = \frac{n}{n^2+1}$, $b_n = \frac{n}{n^2} = \frac{1}{n}$.

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim \frac{n^2}{n^2+1} = 1 \in (0,\infty).$$

By the Limit Comparison Test, Since

$\sum b_n = \sum \frac{1}{n}$ diverges (harmonic series)

So does $\sum a_n = \sum \frac{n}{n^2+1}$.

2. Use the *Ratio/Root Test* to determine if the series converges or diverges.

(*a*) (3 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} e$$

$$= \lim_{n\to\infty} \frac{1}{n+1} e$$

$$= 0 < 1$$

By the Ratio Test,

the series converges

(*b*) (3 pts) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n-1}\right)^n$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n-1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{n}{2n-1}$$

$$= \frac{1}{2} < 1$$

By the Root Test, the series converges.

3. Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Use the *Alternating Series Test* where appropriate.

(a) (4 pts) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$

- $a_n = \frac{(-1)^n}{n\sqrt{n}}$ is alternating and $|a_n| = \frac{1}{n\sqrt{n}} \searrow 0$ (decreases to 0)

So by the Alternating Series Test, the series converges.

- $\sum |a_n| = \sum \frac{1}{n\sqrt{n}} = \sum \frac{1}{n^{3/2}}$ converges by p-series $(p = \frac{3}{2} > 1)$.

So the series in fact converges absolutely.

(b) (4 pts) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{2n+1}}$

- $a_n = \frac{(-1)^{n-1}}{\sqrt{2n+1}}$ is alternating and $|a_n| = \frac{1}{\sqrt{2n+1}} \searrow 0$.

By the AST, the series converges.

- $\sum |a_n| = \sum \frac{1}{\sqrt{2n+1}} \overset{\text{(Limit Comparison)}}{\sim} \sum \frac{1}{\sqrt{2n}} = \sum \frac{1}{\sqrt{2}} \frac{1}{\sqrt{n}}$

Since $\sum \frac{1}{\sqrt{2}} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{2}} \sum \frac{1}{n^{1/2}}$ diverges $(p = \frac{1}{2} \leq 1)$

by the Limit Comparison test, $\sum |a_n|$ diverges.

Therefore the series $\sum a_n$ converges conditionally.

Bonus. (4 pts) Use the *Integral Test* to determine if the series is convergent or divergent.

$$\sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$$

Let $f(x) = \frac{1}{x(\ln x)(\ln \ln x)}$, $x \geq 4$. Then $f(x) \geq 0$, $f(x)$ is decreasing.

Moreover $\int_4^{\infty} f(x)\,dx = \int_4^{\infty} \frac{1}{x(\ln x)(\ln \ln x)}\,dx$

$\overset{u=\ln x}{=} \int_{\ln 4}^{\infty} \frac{1}{u(\ln u)}\,du$

$\overset{v=\ln u}{=} \int_{\ln \ln 4}^{\infty} \frac{1}{v}\,dv$

$= \ln(\infty) - \ln \ln \ln(4)$

$= \infty$ (diverges)

By the Integral Test, $\sum_4^{\infty} f(n) = \sum_4^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$ diverges too.