11.1#2 Let $P = \{R_{ij}\}$ be any partition of the rectangle $R = [a, b] \times [c, d]$. Since $f(x, y) \equiv k$, we have

$$M_{ij} = \sup_{R_{ij}} f = k, \quad m_{ij} = \inf_{R_{ij}} f = k$$

for any i and j. Therefore,

$$U(P, f) = \sum_{ij} M_{ij}|R_{ij}| = k \sum_{ij} |R_{ij}| = k(b-a)(d-c),$$

$$L(P, f) = \sum_{ij} m_{ij}|R_{ij}| = k \sum_{ij} |R_{ij}| = k(b-a)(d-c).$$

Consequently,

$$\overline{\iint}_R f = \inf_P U(P, f) = k(b-a)(d-c),$$

$$\underline{\iint}_R f = \sup_P L(P, f) = k(b-a)(d-c).$$

Since $\overline{\iint}_R f = \underline{\iint}_R f = k(b-a)(d-c)$, by definition, f is Riemann integrable on R with $\iint_R f = k(b-a)(d-c)$.

11.1#6 Let $P = \{R_{ij}\}$ be any partition of the rectangle $R = [0, 1] \times [0, 1]$. For any i and j, since R_{ij} contains a point (x, y) with $x \in \mathbb{Q}$, we have

$$M_{ij} = \sup_{R_{ij}} f = 1;$$

similarly, since R_{ij} contains a point (x, y) with $x \notin \mathbb{Q}$, we have

$$m_{ij} = \inf_{R_{ij}} f = 0.$$

From this we find that

$$U(P, f) = \sum_{ij} M_{ij}|R_{ij}| = \sum_{ij} |R_{ij}| = 1,$$

$$L(P, f) = \sum_{ij} m_{ij}|R_{ij}| = 0,$$

and therefore

$$\overline{\iint}_R f = \inf_P U(P, f) = 1,$$

$$\underline{\iint}_R f = \sup_P L(P, f) = 0.$$

Since $\overline{\iint}_R f \neq \underline{\iint}_R f$, by definition, f is not Riemann integrable on R.

11.2#5 (a) Let's consider three different cases.
Case 1: $y < 1$. In this case we have

$$\overline{\int_0^1} f(x,y)dx = 2 < \underline{\int_0^1} f(x,y)dx = 2y.$$

Therefore $\int_0^1 f(x,y)dx$ does not exist.
Case 2: $y = 1$. In this case we have

$$f(x,y) = 2, \ \forall x \in [0,1].$$

So $\int_0^1 f(x,y)dx = 2$ (exists).
Case 3: $y > 1$. In this case we have

$$\overline{\int_0^1} f(x,y)dx = 2y > \underline{\int_0^1} f(x,y)dx = 2.$$

Therefore $\int_0^1 f(x,y)dx$ does not exist.
Since $\int_0^1 f(x,y)dx$ exists only when $y = 1$, the iterated integral $\int_0^2 \left[\int_0^1 f(x,y)dx\right] dy$ DNE.
(b) Let's consider two different cases.
Case 1: $x \notin \mathbb{Q}$. In this case we have

$$f(x,y) = 2, \ \forall y \in [0,2].$$

So $\int_0^2 f(x,y)dy = 4$ (exists).
Case 2: $x \in \mathbb{Q}$. In this case we have

$$f(x,y) = 2y, \ \forall y \in [0,2].$$

So $\int_0^2 f(x,y)dy = \int_0^2 2ydy = 4$ (exists).
Therefore the iterated integral $\int_0^1 \left[\int_0^2 f(x,y)dy\right] dx = \int_0^1 4dx = 4$ (exists).
(c) We show that f is not Riemann integrable on $R = [0,1] \times [0,2]$. If f were Riemann integrable, then by Fubini's theorem, we would have

$$\int_0^1 \left[\overline{\int_0^2} f(x,y)dy\right] dx = \int_0^2 \left[\overline{\int_0^1} f(x,y)dx\right] dy.$$

By the computation above, this leads to

$$4 = \int_0^2 \max(2y,2)\, dy.$$

Since

$$\int_0^2 \max(2y,2)\, dy = \int_0^1 2dy + \int_1^2 2ydy = 5 \neq 4,$$

we obtain a contradiction.

11.2#7(a) Since $f(x)$ and $g(y)$ are both Riemann integrable, it can be shown that $F(x,y) := f(x)g(y)$ is Riemann integrable on $R = [a,b] \times [c,d]$ (show this). By Fubini's theorem, we then have (note that the inner integral always exists)

$$\begin{aligned}
\iint_R f(x)g(y)dxdy &= \int_c^d \left[\int_a^b f(x)g(y)dx\right] dy \\
&= \int_c^d g(y) \left[\int_a^b f(x)dx\right] dy \\
&= \left[\int_a^b f(x)dx\right] \int_c^d g(y)dy \\
&= \left[\int_a^b f(x)dx\right] \left[\int_c^d g(y)dy\right].
\end{aligned}$$

This proves the identity.