7.1#1(e) Since

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2} \neq 0,$$

by the Divergence Test, the series

$$\sum_{k=1}^{\infty} \frac{k}{2k+1}$$

diverges.

7.1#1(f) Notice that,

$$a_k = \frac{2}{k^2+k} = \frac{1}{k} - \frac{1}{k+2}.$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{2}{k^2+k} &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right) \\ &= \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots \\ &= 1 + \frac{1}{2} \\ &= \frac{3}{2}.\end{aligned}$$

7.1#4(a) The statement is false. Consider

$$a_k = \frac{1}{k}, \quad b_k = -\frac{1}{k}.$$

Then both $\sum a_k$ and $\sum b_k$ diverge, but

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} 0$$

converges.

7.1#4(c) The statement is true. Assume otherwise that

$$\sum_{k=1}^{\infty} (a_k + b_k)$$

converges. Then

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} [(a_k + b_k) - a_k] = \sum_{k=1}^{\infty} (a_k + b_k) - \sum_{k=1}^{\infty} a_k$$

converges, since $\sum a_k$ converges. But this contradicts the assumption that $\sum b_k$ diverges. Therefore $\sum(a_k + b_k)$ must be divergent.

7.1#17 We show by definition that

$$\lim_{n\to\infty} S_n = 0.$$

Observe that

$$S_{2k} = (1-1) + \left(\frac{1}{2} - \frac{1}{2}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k}\right) = 0.$$

Therefore,

$$\lim_{k\to\infty} S_{2k} = 0.$$

On the other hand,

$$S_{2k+1} = (1-1) + \left(\frac{1}{2} - \frac{1}{2}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k}\right) + \frac{1}{k+1} = \frac{1}{k+1}.$$

So we also have

$$\lim_{k\to\infty} S_{2k+1} = 0.$$

Combining these we conclude

$$\lim_{n\to\infty} S_n = 0,$$

as desired.

7.2#1(k) Notice that

$$\ln k \geq 1,\ \forall k \geq 3.$$

Therefore,

$$\frac{\ln k}{k} \geq \frac{1}{k},\ \forall k \geq 3.$$

Since

$$\sum_{k=3}^{\infty} \frac{1}{k}$$

diverges. By the Comparison Test, we have that

$$\sum_{k=3}^{\infty} \frac{\ln k}{k}$$

diverges.

7.2#1(s) Consider

$$b_k = \frac{\sqrt{k}}{\sqrt{k^3}} = \frac{1}{k}.$$

Then

$$\lim_{k\to\infty}\frac{b_k}{a_k}=\frac{\sqrt{k^3}+1}{\sqrt{k^3}}=1<\infty.$$

By the Limit Comparison Test, since $\sum b_k$ diverges, we have

$$\sum_{k=1}^{\infty} a_k=\sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{k^3}+1}$$

diverges.

7.2#5 Since

$$\sum_{k=1}^{\infty} a_k$$

converges, we must have

$$\lim_{k\to\infty} a_k=0.$$

This implies that $\{a_k\}$ is bounded, i.e. there exists $M>0$ such that

$$|a_k|=a_k\leq M,\ \forall k\geq 1.$$

From this we deduce that

$$0\leq a_k^2=a_ka_k\leq Ma_k,\ \forall k\geq 1.$$

By the Comparison Test, since $\sum Ma_k$ converges,

$$\sum_{k=1}^{\infty} a_k^2$$

converges as well. This completes the proof.