8.1#1(f) Since

$$\left|\frac{\sin(nx)}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}} \to 0,$$

we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin(nx)}{\sqrt{n}} = 0$$

for all x.

8.1#1(i) For $x \in [0, 1)$, we have

$$0 \leq f_n(x) = \frac{x^n}{1 + x^n} \leq x^n \to 0, \ n \to \infty.$$

Therefore

$$f(x) = \lim_{n\to\infty} f_n(x) = 0, \ \forall x \in [0, 1)$$

For $x = 1$, we have

$$f_n(1) = \frac{1}{1+1} = \frac{1}{2}, \ \forall n.$$

Therefore

$$f(1) = \lim_{n\to\infty} f_n(1) = \frac{1}{2}.$$

For $x \in (1, \infty)$, we have

$$f_n(x) = \frac{x^n}{1 + x^n} = \frac{1}{x^{-n} + 1} \to 1, \ n \to \infty.$$

Therefore

$$f(x) = \lim_{n\to\infty} f_n(x) = 1, \ \forall x \in (1, \infty)$$

8.2#2 Given $\varepsilon > 0$, since f_n converges uniformly to f on D, there exists n_1^* such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \ \forall x \in D, \ n \geq n_1^*.$$

Similarly, since g_n converges uniformly to g on D, there exists n_2^* such that

$$|g_n(x) - g(x)| < \frac{\varepsilon}{2}, \ \forall x \in D, \ n \geq n_2^*.$$

Let $n^* = \max(n_1^*, n_2^*)$. Then for all $n \geq n^*$, $x \in D$, we have

$$\begin{aligned} |(f_n(x) + g_n(x)) - (f(x) + g(x))| &= |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

By definition, this shows $f_n + g_n$ converges uniformly to $f + g$ on D.

8.2#3(a) Let $\varepsilon = 1$. Since f_n converges uniformly to f on D, there exists n^* such that

$$|f_n(x) - f(x)| < \varepsilon = 1, \ \forall x \in D, \ n \geq n^*.$$

In particular, taking $n = n^*$, we have

$$|f_{n^*}(x) - f(x)| < 1, \ \forall x \in D.$$

Since f_{n^*} is bounded, there exists $M > 0$ such that

$$|f_{n^*}(x)| \leq M, \ \forall x \in D.$$

Therefore, for all $x \in D$, we have

$$\begin{aligned} |f(x)| &= |(f(x) - f_{n^*}(x)) + f_{n^*}(x)| \\ &\leq |f(x) - f_{n^*}(x)| + |f_{n^*}(x)| \\ &\leq 1 + M. \end{aligned}$$

This shows f is bounded on D.

8.2#6(a) ($\Rightarrow$) Given $\varepsilon > 0$, since f_n converges uniformly to f on D, there exists n^* such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \ \forall x \in D, \ n \geq n^*.$$

Taking sup over $x \in D$, we obtain that, for all $n \geq n^*$,

$$\sup_{x \in D} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

That is,

$$M_n < \varepsilon, \ \forall n \geq n^*.$$

By definition, this shows

$$\lim_{n \to \infty} M_n = 0.$$

($\Leftarrow$) Given $\varepsilon > 0$, since

$$\lim_{n \to \infty} M_n = 0.$$

there exists n^* such that

$$M_n < \varepsilon, \ \forall n \geq n^*.$$

In particular, for all $n \geq n^*$, $x \in D$, we have

$$|f_n(x) - f(x)| \leq \sup_{x \in D} |f_n(x) - f(x)| = M_n < \varepsilon.$$

By definition, this shows f_n converges uniformly to f on D.