8.3#1 Note that, for $x \geq 2$, we have

$$\frac{x^n}{1+x^{2n}} \leq \frac{x^n}{x^{2n}} = \frac{1}{x^n} \leq \frac{1}{2^n} \to 0.$$

Therefore

$$f_n(x) = \frac{x^n}{1+x^{2n}}$$

converges uniformly to 0 on $[2,5]$. Clearly, each f_n is continuous on $[2,5]$. So, by Theorem 8.3.3, we have

$$\lim_{n\to\infty} \int_2^5 f_n(x)\,dx = \int_2^5 \left(\lim_{n\to\infty} f_n(x)\right)\,dx = 0.$$

8.3#6 Since f_n is continuous and converges uniformly to f on D, by Theorem 8.3.1, f is continuous on D. In particular, for any given $\varepsilon > 0$, there exists n_1^* such that

$$|f(x_n) - f(c)| < \frac{\varepsilon}{2},\ \forall n \geq n_1^*.$$

On the other hand, by uniform convergence, there exists n_2^* such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2},\ \forall n \geq n_2^*,\ \forall x \in D.$$

In particular, letting $x = x_n$, this implies

$$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2},\ \forall n \geq n_2^*.$$

Therefore, for all $n \geq n^* = \max(n_1^*, n_2^*)$, we have

$$\begin{aligned} |f_n(x_n) - f(c)| &= |f_n(x_n) - f(x_n) + f(x_n) - f(c)| \\ &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(c)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\lim_{n\to\infty} f_n(x_n) = f(c),$$

as desired.

8.4#14 Let

$$F_n = \sum_{k=1}^{n} f_k.$$

Then F_n is continuously differentiable and converges pointwise to $F = \sum_{k=1}^{\infty} f_k$. Moreover, by the assumption we have that F_n' converges uniformly to

$$\lim_{n\to\infty} F_n' = \lim_{n\to\infty} \sum_{k=1}^{n} f_k' = \sum_{k=1}^{\infty} f_k'.$$

By Theorem 8.3.5, this implies that $F = \sum_{k=1}^{\infty} f_k$ is differentiable and moreover

$$F' = \sum_{k=1}^{\infty} f_k'.$$