7.4#2(d) Note that

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k}{4k^2-3}$$

is an alternating series. To show that it converges, by the Alternating Series Test, we need to show

$$\lim_{k\to\infty}\frac{k}{4k^2-3}=0$$

(which is obvious) and that

$$\frac{k+1}{4(k+1)^2-3}\leq\frac{k}{4k^2-3}.$$

The last inequality is equivalent to

$$(k+1)(4k^2-3)\leq k\left(4(k+1)^2-3\right)$$

which is

$$4k^3+4k^2-3k-3\leq 4k^3+8k^2+k,$$

or,

$$-3\leq 4k^2+4k.$$

But this is clearly true since $k\geq 1$.

7.4#4 For any fixed n, by the triangle inequality, we have

$$\left|\sum_{k=1}^{n}a_k\right|\leq\sum_{k=1}^{n}|a_k|.$$

Taking n to infinity, we obtain

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n}a_k\right|\leq\lim_{n\to\infty}\sum_{k=1}^{n}|a_k|,$$

or

$$\left|\lim_{n\to\infty}\sum_{k=1}^{n}a_k\right|\leq\lim_{n\to\infty}\sum_{k=1}^{n}|a_k|.$$

By the definition of the value of series, this means

$$\left|\sum_{k=1}^{\infty}a_k\right|\leq\sum_{k=1}^{\infty}|a_k|,$$

which is the desired inequality.

8.1#1(f) Since

$$\left|\frac{\sin(nx)}{\sqrt{n}}\right|\leq\frac{1}{\sqrt{n}}\to 0,$$

we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin(nx)}{\sqrt{n}} = 0$$

for all x.

8.1#1(i) For $x \in [0, 1)$, we have

$$0 \le f_n(x) = \frac{x^n}{1 + x^n} \le x^n \to 0,\ n \to \infty.$$

Therefore

$$f(x) = \lim_{n\to\infty} f_n(x) = 0,\ \forall x \in [0, 1)$$

For $x = 1$, we have

$$f_n(1) = \frac{1}{1+1} = \frac{1}{2},\ \forall n.$$

Therefore

$$f(1) = \lim_{n\to\infty} f_n(1) = \frac{1}{2}.$$

For $x \in (1, \infty)$, we have

$$f_n(x) = \frac{x^n}{1 + x^n} = \frac{1}{x^{-n} + 1} \to 1,\ n \to \infty.$$

Therefore

$$f(x) = \lim_{n\to\infty} f_n(x) = 1,\ \forall x \in (1, \infty).$$