8.2#2 Given $\varepsilon > 0$, since f_n converges uniformly to f on D, there exists n_1^* such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \ \forall x \in D, \ n \geq n_1^*.$$

Similarly, since g_n converges uniformly to g on D, there exists n_2^* such that

$$|g_n(x) - g(x)| < \frac{\varepsilon}{2}, \ \forall x \in D, \ n \geq n_2^*.$$

Let $n^* = \max(n_1^*, n_2^*)$. Then for all $n \geq n^*$, $x \in D$, we have

$$\begin{aligned} |(f_n(x) + g_n(x)) - (f(x) + g(x))| &= |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

By definition, this shows $f_n + g_n$ converges uniformly to $f + g$ on D.

8.2#3(a) Let $\varepsilon = 1$. Since f_n converges uniformly to f on D, there exists n^* such that

$$|f_n(x) - f(x)| < \varepsilon = 1, \ \forall x \in D, \ n \geq n^*.$$

In particular, taking $n = n^*$, we have

$$|f_{n^*}(x) - f(x)| < 1, \ \forall x \in D.$$

Since f_{n^*} is bounded, there exists $M > 0$ such that

$$|f_{n^*}(x)| \leq M, \ \forall x \in D.$$

Therefore, for all $x \in D$, we have

$$\begin{aligned} |f(x)| &= |(f(x) - f_{n^*}(x)) + f_{n^*}(x)| \\ &\leq |f(x) - f_{n^*}(x)| + |f_{n^*}(x)| \\ &\leq 1 + M. \end{aligned}$$

This shows f is bounded on D.

8.2#6(a) ($\Rightarrow$) Given $\varepsilon > 0$, since f_n converges uniformly to f on D, there exists n^* such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \ \forall x \in D, \ n \geq n^*.$$

Taking sup over $x \in D$, we obtain that, for all $n \geq n^*$,

$$\sup_{x \in D} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

That is,

$$M_n < \varepsilon, \ \forall n \geq n^*.$$

By definition, this shows

$$\lim_{n\to\infty} M_n = 0.$$

($\Leftarrow$) Given $\varepsilon > 0$, since

$$\lim_{n\to\infty} M_n = 0.$$

there exists n^* such that

$$M_n < \varepsilon, \ \forall n \geq n^*.$$

In particular, for all $n \geq n^*$, $x \in D$, we have

$$|f_n(x) - f(x)| \leq \sup_{x\in D} |f_n(x) - f(x)| = M_n < \varepsilon.$$

By definition, this shows f_n converges uniformly to f on D.

8.3#1 Note that, for $x \geq 2$, we have

$$\frac{x^n}{1+x^{2n}} \leq \frac{x^n}{x^{2n}} = \frac{1}{x^n} \leq \frac{1}{2^n} \to 0.$$

Therefore

$$f_n(x) = \frac{x^n}{1+x^{2n}}$$

converges uniformly to 0 on $[2,5]$. Clearly, each f_n is continuous on $[2,5]$. So, by Theorem 8.3.3, we have

$$\lim_{n\to\infty} \int_2^5 f_n(x)\, dx = \int_2^5 \left(\lim_{n\to\infty} f_n(x) \right) dx = 0.$$

8.3#6 Since f_n is continuous and converges uniformly to f on D, by Theorem 8.3.1, f is continuous on D. In particular, for any given $\varepsilon > 0$, there exists n_1^* such that

$$|f(x_n) - f(c)| < \frac{\varepsilon}{2}, \ \forall n \geq n_1^*.$$

On the other hand, by uniform convergence, there exists n_2^* such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \ \forall n \geq n_2^*, \ \forall x \in D.$$

In particular, letting $x = x_n$, this implies

$$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}, \ \forall n \geq n_2^*.$$

Therefore, for all $n \geq n^* = \max(n_1^*, n_2^*)$, we have

$$\begin{aligned} |f_n(x_n) - f(c)| &= |f_n(x_n) - f(x_n) + f(x_n) - f(c)| \\ &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(c)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\lim_{n\to\infty} f_n(x_n) = f(c),$$

as desired.