10.1#6 Assuming $B_1, \cdots, B_n$ are open sets. To show that $\cap_{k=1}^n B_k$ is open, given any $x \in \cap_{k=1}^n B_k$, we show that there exists $r > 0$ such that

$$B(x, r) \subset \cap_{k=1}^n B_k$$

(thus x is an interior point of $\cap_{k=1}^n B_k$). For any $k = 1, \cdots, n$, since $x \in B_k$ and B_k is open, there must exist $r_k > 0$ such that

$$B(x, r_k) \subset B_k.$$

Now letting

$$r = \min_{1 \le k \le n} r_k,$$

we have $r > 0$ and

$$B(x, r) \subset B(x, r_k) \subset B_k, \text{ for any } k = 1, \cdots, n.$$

Therefore

$$B(x, r) \subset \cap_{k=1}^n B_k,$$

as desired.

10.1#8 (a) The set

$$D = \{x \in \mathbb{R}^2 : 0 < \|x\| \le 1\}$$

is neither closed nor open. It is not open because $(1, 0) \in D$ but is not an interior point. It is not closed because $(0, 0)$ is an accumulation point but not included in D.
(b) $D = \mathbb{R}^2$ is both open and closed.

10.2#4 (c) Consider the sequence

$$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right), \quad n = 1, 2, \cdots.$$

Clearly we have

$$\lim_{n \to \infty} (x_n, y_n) = (0, 0).$$

By the definition of f,

$$f(x_n, y_n) = \frac{1}{2}, \text{ for all } n = 1, 2, \cdots.$$

In particular,

$$\lim_{n \to \infty} f(x_n, y_n) \neq f(0, 0) = 0.$$

This shows f is discontinuous at $(0, 0)$.
(b) Consider two different cases.
Case 1: $b = 0$. In this case we have $g(x) = 0$, $\forall x \in \mathbb{R}$. Thus $g(x)$ is continuous at $x = 0$.
Case 2: $b \neq 0$. In this case we have

$$g(x) = \frac{bx}{b^2 + x^2}$$

which is clearly continuous at $x = 0$ since $b \neq 0$. Part (c) is similar to part (b).

10.2#10(a) By Theorem 10.2.7, to show that fg is continuous at (a, b), it suffices to show that for any sequence $(x_n, y_n) \in D$ with

$$\lim_{n\to\infty} (x_n, y_n) = (a, b),$$

we have

$$\lim_{n\to\infty} (fg)(x_n, y_n) = (fg)(a, b).$$

Indeed, by the continuity of f and g at (a, b), we have

$$\lim_{n\to\infty} f(x_n, y_n) = f(a, b), \quad \lim_{n\to\infty} g(x_n, y_n) = g(a, b).$$

Therefore

$$\lim_{n\to\infty} f(x_n, y_n) \cdot g(x_n, y_n) = f(a, b) \cdot g(a, b).$$

This shows

$$\lim_{n\to\infty} (fg)(x_n, y_n) = (fg)(a, b),$$

as desired.

10.2#11(b) Assume to the contrary that f is unbounded on D. Then for any positive integer n, since f is not bounded by n, there must exist $(x_n, y_n) \in D$ such that

$$|f(x_n, y_n)| > n.$$

Note that the sequence $\{(x_n, y_n)\}$ is bounded in $\mathbb{R}^2$ since D is bounded. By the Bolzano-Weierstrass theorem, $\{(x_n, y_n)\}$ has a convergent subsequence $\{(x_{n_k}, y_{n_k})\}$, say

$$\lim_{k\to\infty} (x_{n_k}, y_{n_k}) = (\bar{x}, \bar{y}).$$

Since D is closed, we must have

$$(\bar{x}, \bar{y}) \in D.$$

On the other hand, since f is continuous on D, we have

$$\lim_{k\to\infty} f(x_{n_k}, y_{n_k}) = f(\bar{x}, \bar{y}).$$

However, this contradicts the property of (x_n, y_n):

$$|f(x_{n_k}, y_{n_k})| > n_k \to \infty, \quad k \to \infty.$$

Thus f must be bounded on D.