10.4#7 (a) For $(x, y) \neq (0, 0)$, we have

$$\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = |x| \frac{|y|}{\sqrt{x^2 + y^2}} \leq |x|$$

where the last bound is because

$$\frac{|y|}{\sqrt{x^2 + y^2}} = \frac{\sqrt{0^2 + y^2}}{\sqrt{x^2 + y^2}} \leq 1.$$

From this it is clear that

$$\lim_{(x,y)\to(0,0)} \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \leq \lim_{(x,y)\to(0,0)} |x| = 0.$$

Thus the continuity of f at $(0, 0)$.
(b) Notice that, by the definition of f, we have for all x and y,

$$f(x, 0) = 0, \quad f(0, y) = 0.$$

It follows that

$$f_x(x, 0) = 0, \quad f_y(0, y) = 0.$$

In particular,

$$f_x(0, 0) = 0, \quad f_y(0, 0) = 0.$$

(c) By the definition of differentiability, if f were differentiable at $(0, 0)$, then

$$f(x, y) = f_x(0, 0)\, x + f_y(0, 0)\, y + \varepsilon\sqrt{x^2 + y^2}$$

(note that $f(0, 0) = 0$), where

$$\lim_{(x,y)\to(0,0)} \varepsilon = 0.$$

However, according to part (b) we have

$$f_x(0, 0) = 0, \quad f_y(0, 0) = 0.$$

Therefore

$$f(x, y) = \varepsilon\sqrt{x^2 + y^2},$$

and so,

$$\lim_{(x,y)\to(0,0)} \varepsilon = \lim_{(x,y)\to(0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = 0.$$

By the definition of f, this means

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = 0.$$

However, by Exercise 10.1.4, this limit does not hold. Thus f is not differentiable at $(0, 0)$.

10.4#14 Since f and g are both differentiable at (x_0, y_0), we have

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\,(x - x_0) + f_y(x_0,y_0)\,(y - y_0) + \varepsilon\sqrt{(x-x_0)^2 + (y-y_0)^2},$$

$$g(x,y) = g(x_0,y_0) + g_x(x_0,y_0)\,(x - x_0) + g_y(x_0,y_0)\,(y - y_0) + \eta\sqrt{(x-x_0)^2 + (y-y_0)^2},$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0, \qquad \lim_{(x,y)\to(x_0,y_0)} \eta = 0.$$

For simplicity we will write these as

$$f = f_0 + a\,\Delta x + b\,\Delta y + \varepsilon\sqrt{\Delta x^2 + \Delta y^2},$$

$$g = g_0 + c\,\Delta x + d\,\Delta y + \eta\sqrt{\Delta x^2 + \Delta y^2}.$$

(a) We can write

$$f \pm g = (f_0 \pm g_0) + (a \pm c)\,\Delta x + (b \pm d)\,\Delta y + (\varepsilon \pm \eta)\sqrt{\Delta x^2 + \Delta y^2},$$

where clearly

$$\lim_{(x,y)\to(x_0,y_0)} (\varepsilon + \eta) = 0.$$

This shows $f \pm g$ are differentiable at (x_0, y_0).
(b) We can write

$$\begin{aligned} fg &= \left(f_0 + a\,\Delta x + b\,\Delta y + \varepsilon\sqrt{\Delta x^2 + \Delta y^2}\right)\left(g_0 + c\,\Delta x + d\,\Delta y + \eta\sqrt{\Delta x^2 + \Delta y^2}\right) \\ &= f_0 g_0 + (a g_0 + f_0 c)\Delta x + (b g_0 + f_0 d)\Delta y + E \end{aligned}$$

where

$$\begin{aligned} E = {} & (a\Delta x + b\,\Delta y)(c\Delta x + d\,\Delta y) \\ & + \eta\,(f_0 + a\,\Delta x + b\,\Delta y)\sqrt{\Delta x^2 + \Delta y^2} \\ & + \varepsilon\,(g_0 + c\,\Delta x + d\,\Delta y)\sqrt{\Delta x^2 + \Delta y^2} \\ & + \varepsilon\eta\,(\Delta x^2 + \Delta y^2). \end{aligned}$$

It can be shown that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E}{\sqrt{\Delta x^2 + \Delta y^2}} = 0,$$

from which it follows that fg is differentiable at (x_0, y_0).