Corrections are welcome.

2011Aug#1. This problem is the same as **2006Aug#2**.

2011Aug#2. (Note that we can not use Fubini's theorem here.) By Tonelli's theorem (justify applicability), if $\beta \neq 1$,

$$\begin{aligned} &\int\int_{[0,1]\times[0,1]} \frac{1}{(x+y^\alpha)^\beta} dxdy \\ = &\int_0^1 \Big(\int_0^1 \frac{1}{(x+y^\alpha)^\beta} dx \Big) dy \\ = &\int_0^1 \frac{1}{1-\beta} \Big((1+y^\alpha)^{1-\beta} - y^{\alpha(1-\beta)} \Big) dy. \end{aligned}$$

Since $(1+y^\alpha)^{1-\beta} \approx 1$, the last line is finite if and only if

$$\int_0^1 y^{\alpha(1-\beta)} dy < \infty.$$

By the fundamental theorem of calculus, this holds if and only if

$$\alpha(1-\beta) > -1.$$

If $\beta = 1$, then the same computation leads to

$$\int_0^1 \Big(\log(1+y^\alpha) - \log(y^\alpha) \Big) dy$$

whose finiteness is determined by

$$\int_0^1 \log(y^\alpha) dy$$

which is always finite.

2011Aug#3. (i) B is not compact in $(E, \|\cdot\|_1)$ by Riesz's lemma. More precisely, consider vectors

$$e_n = (0, \cdots, 0, 1, 0, \cdots)$$

where 1 occurs in the nth coordinate. Then clearly the sequence $\{e_n\}$ is contained in B. However $\{e_n\}$ has no convergent subsequence, as $\|e_i - e_j\| = 1$ whenever $i \neq j$.

(ii) Denote

$$E_n = \{x = (x_1, x_2, \cdots) : x_{n+1} = x_{n+2} = \cdots = 0\}$$

and

$$B_n = B \cap E_n.$$

Then B_n is a compact set since it is bounded and closed in the finite dimensional space E_n. Moreover,

$$B \subset U\big(B_n, \frac{1}{n+1}\big),$$

meaning that for every $x \in B$, there exists $x^{(n)} \in B_n$ such that

$$\|x - x^{(n)}\|_2 \le \frac{1}{n+1}.$$

Such an $x^{(n)}$ can be taken to be the projection of x to the first n coordinates. Now one can prove that B is completely bounded using ϵ-nets of B_n for large enough n. Since B is clearly closed in the Banach space $(E, \|\cdot\|_2)$, B is compact.

2011Aug#4. Denote by $\mathcal{L}$ the Lebesgue σ-algebra. Then $f(x) - f(y)$ is $\mathcal{L} \times \mathcal{L}$-measurable. By Fubini or Tonelli's theorem (justify applicability),

$$\int\int_{[0,1]\times[0,1]} |f(x) - f(y)| dx dy = \int_0^1 \Big(\int_0^1 |f(x) - f(y)| dx\Big) dy < \infty$$

Write

$$F(y) = \int_0^1 |f(x) - f(y)| dx.$$

Then $F(y) < \infty$ for a.e. $y \in [0,1]$. Fix such an y, we see that

$$\int_0^1 |f(x) - c| dx < \infty$$

where $c = f(y)$. This implies $f \in L^1[0,1]$.

2011Aug#5. First, notice that since $f_n \to f$ in Lebesgue measure, there exists a subsequence n_k such that $f_{n_k} \to f$ a.e. on $[0.1]$. By Fatou's lemma (applied to $|f_n|^2$), we see that $\|f\|_{L^2[0,1]} \le 1$.

Given $\epsilon > 0$, write

$$E_n = \{x \in [0,1] : |f_n(x) - f(x)| > \epsilon\}.$$

Then

$$\begin{aligned}
&\Big|\int_0^1 (f_n(x) - f(x)) g(x) dx\Big| \\
&\le \Big|\int_{E_n} (f_n(x) - f(x)) g(x) dx\Big| + \Big|\int_{E_n^c} (f_n(x) - f(x)) g(x) dx\Big| \\
&\le \Big(\int_{E_n} |f_n(x) - f(x)|^2 dx|\Big)^{1/2} \Big(\int_{E_n} |g(x)|^2 dx|\Big)^{1/2} + \int_{E_n^c} |f_n(x) - f(x)||g(x)| dx \\
&\le (\|f_n\|_{L^2[0,1]} + \|f\|_{L^2[0,1]}) \Big(\int_{E_n} |g(x)|^2 dx|\Big)^{1/2} + \epsilon \int_{E_n^c} |g(x)| dx
\end{aligned}$$

$$\le 2\Big(\int_{E_n} |g(x)|^2 dx|\Big)^{1/2} + \epsilon \|g\|_{L^2[0,1]}.$$

Since $|E_n| \to 0$ as $n \to \infty$, we have

$$\Big(\int_{E_n} |g(x)|^2 dx|\Big)^{1/2} \to 0.$$

Thus

$$\limsup_{n\to\infty} \Big|\int_0^1 (f_n(x) - f(x))g(x)dx\Big| \le \epsilon \|g\|_{L^2[0,1]}.$$

Since $\epsilon > 0$ is arbitrary, we conclude the proof.

2011Aug#6. (i) Notice that by

$$\cos(nx + t_n) = \cos(nx)\cos(t_n) - \sin(nx)\sin(t_n),$$

we have

$$\begin{aligned}&\int_E \cos(nx + t_n)dx \\ &= \cos(t_n)\int_0^{2\pi} \chi_E(x)\cos(nx)dx - \sin(t_n)\int_0^{2\pi} \chi_E(x)\sin(nx)dx \\ &\to 0\end{aligned}$$

by the Riemann-Lebesgue lemma.

(ii) This is the same problem as **2010Jan#5**.

2011Aug#7R. By the condition, we have $f_n(x) \to f(x)$ for all $x \in [0,1]$. In particular, for any $\epsilon > 0$.

$$[0,1] = \bigcup_{p\ge 1} \bigcap_{m,n\ge p} \{x \in [0,1] : |f_n(x) - f_m(x)| \le \epsilon/2\}$$

Now set

$$F_p = \bigcap_{m,n\ge p} \{x \in [0,1] : |f_n(x) - f_m(x)| \le \epsilon/2\}.$$

Then F_p is closed set by the continuity of f_n.

We claim that there exists p such that F_p contains an interval. Indeed, if this is not true then F_p is a nowhere dense. But

$$[0,1] = \bigcup_{p\ge 1} F_p$$

which is then a set of first category, contradicting the Baire category theorem since $[0,1]$ is complete.

Now suppose $(a,b) \subset F_p$ for some fixed p. Then for any $x \in (a,b)$,

$$|f_n(x) - f_m(x)| \le \epsilon/2.$$

Let $m \to \infty$, we see that

$$|f_n(x) - f(x)| \le \epsilon/2 < \epsilon.$$

This completes the proof.

2011Aug#8R. (i) We claim that the set

$$B_\lambda = \{(x_1, x_2, \cdots) \in \ell^2(\mathbb{N}) : |x_n| \leq \lambda_n, \forall n\}$$

is compact in $\ell^2(\mathbb{N})$ if and only if

$$\lambda = (\lambda_1, \lambda_2, \cdots) \in \ell^2(\mathbb{N}).$$

Suppose B_λ is compact, then in particular B_λ is bounded, i.e. there exists $C > 0$ such that

$$\|x\|_{\ell^2(\mathbb{N})} \leq C$$

for all $x \in B_\lambda$. For integer $N \geq 1$, let

$$\lambda^{(N)} = (\lambda_1, \cdots, \lambda_N, 0, \cdots).$$

Then $\lambda^{(N)} \in B_\lambda$ and thus

$$\|\lambda^{(N)}\|_{\ell^2(\mathbb{N})} \leq C.$$

Let $N \to \infty$, we see that $\|\lambda\|_{\ell^2(\mathbb{N})} \leq C$.

Suppose $\lambda \in \ell^2(\mathbb{N})$. Let

$$E_N = \{(x_1, x_2, \cdots) \in \ell^2(\mathbb{N}) : x_n = 0, \forall n > N\}$$

and

$$B_\lambda^{(N)} = B_\lambda \cap E_N.$$

Each $B_\lambda^{(N)}$ is compact since it is closed and bounded in a finite dimensional space. Now because

$$B_\lambda^{(N)} \to B_\lambda$$

uniformly, we can conclude that B_λ is also compact using complete boundedness (B_λ is clearly closed in the Banach space $\ell^2(\mathbb{N})$).

(ii) We claim that the set

$$B_\mu = \{(x_1, x_2, \cdots) \in \ell^2(\mathbb{N}) : \sum_n \frac{|x_n|^2}{\mu_n^2} \leq 1\}$$

is compact in $\ell^2(\mathbb{N})$ if and only if

$$\mu = (\mu_1, \mu_2, \cdots) \in c_0,$$

i.e. $\mu_n \to 0$ as $n \to \infty$.

Suppose $\mu_n \to 0$ as $n \to \infty$. Notice that $x \in B_\mu$ implies

$$\sum_{n \geq N} |x_n|^2 \leq \sup_{n \geq N} \mu_n^2 \to 0, \text{ as } N \to \infty.$$

Thus arguing as above, one can approximate B_μ uniformly by finite dimensional compact sets and hence is compact (B_μ is clearly closed).

Suppose B_μ is compact, we now show that $\mu_n \to 0$ as $n \to \infty$. Assume for contradiction that this is not true. Then there exists $\delta > 0$ and a subsequence n_k such that $\mu_{n_k}^2 \geq \delta^2$. If we set

$$x^{(k)} = \delta e_{n_k},$$

where e_{n_k} is the indicator of the n_kth coordinate. Then $x^{(k)} \in B_\mu$, yet

$$\|x^{(k)} - x^{(l)}\|_{\ell^2(\mathbb{N})} = \sqrt{2}\delta.$$

This shows B_μ is not compact, contradiction.

2011Aug#9R. This is essentially the same problem as **2006Aug#7R(a)**, or **2008Aug#8R**.

2011Aug#7C. Consider $D_1 = A \backslash (1, 2)$. Since D_1 is simply connected, $u(z)$ has a harmonic conjugate $v_1(z)$ on D_1. Denote $g_1 = u + iv_1$, then g_1 is analytic on D_1. Let v_2 be the harmonic conjugate of u on $D_2 = A \backslash (-2, -1)$ which agrees with v_1 in $A_- = A \cap \{z : \mathrm{Im} z < 0\}$. Write $g_2 = u + iv_2$. Then on $A_+ = A \cap \{z : \mathrm{Im} z > 0\}$, g_1 and g_2 have the same real part, hence differ only by a purely imaginary constant, i.e. $g_2 - g_1 = i\beta$ on A_+. Now fix a branch of $(\log z)_1$ on D_1, and let

$$f(z) = g_1(z) - \frac{\beta}{2\pi}(\log z)_1.$$

Then clearly f is still analytic on D_1. On D_2, let $(\log z)_2$ be the branch which agrees with $(\log z)_1$ on A_-. Then on D_2, we have

$$f(z) = g_2(z) - \frac{\beta}{2\pi}(\log z)_2.$$

This shows that $f(z)$ is analytic on A, and we have

$$\mathrm{Re} f(z) = u(z) - \frac{\beta}{2\pi} \log |z|.$$

2011Aug#8C. Let Γ be the contour consisting of $\{z : |z| = 2, \mathrm{Re} z \leq 0\}$ and $\{it : -2 \leq t \leq 2\}$. Then on $\mathrm{Re} z \leq 0$, $z^4 + z - 2$ has no zero on or outside Γ. Moreover, for any $0 < \epsilon < 1$, we have

$$|z^4 + z - 2| > (1 - \epsilon)|z|$$

on Γ. By Rouché's theorem, the number of zeros of $z^4 + z - 2$ inside Γ is the same as the number of zeros of $z^4 + \epsilon z - 2$ inside Γ. For small enough ϵ, by Rouché's theorem the zeros of $z^4 + \epsilon z - 2$ are close to the zeros of $z^4 - 2$ which are given by $-\sqrt[4]{2}$, $\sqrt[4]{2}i$, $-\sqrt[4]{2}i$, $\sqrt[4]{2}$. By inspection, adding ϵz to $z^4 - 2$ will make the roots $\sqrt[4]{2}i, -\sqrt[4]{2}i$ moving rightwards. This shows that the answer is 1.

2011Aug#9C. (i) The definition of $\phi(s)$ is independent of $\epsilon \in (0, 2\pi)$ because the integrand

$$\frac{e^{(s-1)\log(-z)}}{e^z - 1}$$

is analytic in $\mathbb{C}\backslash[0, \infty)$ when $\epsilon \in (0, 2\pi)$.

(ii) Fix ϵ and let

$$\Gamma_{\epsilon,N} = \Gamma_{\epsilon,N} \cap \{z : \mathrm{Re} z \leq N\}.$$

Then

$$\phi_N(s) = \frac{1}{2\pi i} \int_{\Gamma_{\epsilon,N}} \frac{e^{(s-1)\log(-z)}}{e^z - 1} dz$$

is analytic in s. Moreover, on any compact set $K \subset \mathbb{C}$, $\phi_N(s)$ converges uniformly to $\phi(s)$. This implies analyticity of $\phi(s)$.

Notice that inside Γ_ϵ

$$\frac{1}{e^z - 1} = \frac{1}{z} \frac{z}{e^z - 1}$$

has a pole at $z = 0$. Hence

$$\begin{aligned} \phi(n) &= \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{(-z)^{n-1}}{e^z - 1} dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{1}{z} \frac{z}{e^z - 1} (-z)^{n-1} dz \\ &= \frac{z}{e^z - 1} (-z)^{n-1} \Big|_{z=0} \\ &= \begin{cases} 1 & n = 1 \\ 0 & n \geq 2. \end{cases} \end{aligned}$$