Corrections are welcome.

2012Aug#1. (a) By convergence of the series we can find a strictly increasing sequence $n_k, k = 1, 2, \cdots$ so that

$$|\sum_{j=M}^{N} a_j| \leq 2^{-k}$$

for all $n_k < M \leq N$. Now set

$$b_j = 2^{k/2}$$

for $n_k < j \leq n_{k+1}, k = 1, 2, \cdots$ and $b_j = 1$ for $j \leq n_1$. Then clearly, $\lim_{n\to\infty} b_n = \infty$. Moreover, by Cauchy's test, the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

(b) Wlog assume $b_{n_k} \geq 2^k$. Now set $a_{n_k} = 2^{-k}$ and $a_j = 0$ otherwise.

2012Aug#2. Denote by $F(t)$ the left hand side. Differentiation under the integral (justify) gives

$$F'(t) = \int_0^{\infty} e^{-tx} \sin(x) dx.$$

Integrate by parts twice and move terms, one sees that

$$F'(t) = -\frac{1}{1+t^2}.$$

This means

$$F(t) = C - \arctan t$$

for some constant C. Notice that by Lebesgue DCT (justify),

$$\lim_{t\to\infty} F(t) = 0.$$

On the other hand $\lim_{t\to\infty} \arctan t = \frac{\pi}{2}$. This shows $C = \frac{\pi}{2}$.

2012Aug#3. Suppose $T/2 \leq s \leq T$, then

$$\begin{aligned}
|f(T) - f(s)| &= |\int_s^T f'(t)dt| \\
&= |\int_s^T \frac{1}{\sqrt{t}} \sqrt{t} f'(t) dt| \\
&\leq (\int_s^T \frac{1}{t} dt)^{1/2} (\int_s^T t|f'(t)|^2 dt)^{1/2} \\
&\leq (\int_{T/2}^T \frac{1}{t} dt)^{1/2} (\int_{T/2}^T t|f'(t)|^2 dt)^{1/2}.
\end{aligned}$$

Since

$$\int_{T/2}^T \frac{1}{t} dt = \log 2$$

and
$$\int_{T/2}^{T} t|f'(t)|^2 dt \to 0, \text{ as } T \to \infty,$$
we see that
$$\sup_{T/2 \le t \le T} |f(T) - f(t)| \to 0, \text{ as } T \to \infty.$$
On the other hand,
$$\frac{1}{T/2}\int_{T/2}^{T} f(t)dt = \frac{2}{T}\int_0^T f(t)dt - \frac{1}{T/2}\int_0^{T/2} f(t)dt \to L.$$
Hence
$$f(T) = \frac{1}{T/2}\int_{T/2}^{T} f(t)dt + \frac{1}{T/2}\int_{T/2}^{T} f(T) - f(t)dt \to L.$$

2012Aug#4. These two are standard theorems in real analysis texts.

2012Aug#5. One can take, for example,
$$f_1(x) = 1 - 2x,$$
$$f_2(x) = 2x - 1,$$
$$f_3(x) = \cos(2\pi x),$$
$$f_{3k+l} = f_l.$$
2012Aug#6. By approximation (the standard 3ε-argument), we may assume that f is a trigonometric polynomial. By linearity, we may further assume that $f(x) = e^{2\pi i j x}$ for some $j \in \mathbb{Z}$. Now notice that
$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N} f(kx) &= \frac{1}{N}\sum_{k=1}^{N} e^{2\pi ijkx} \\ &= \frac{1}{N}\sum_{k=1}^{N}\left(e^{2\pi ijx}\right)^k \\ &= \frac{1}{N}\frac{e^{2\pi ij(N+1)x} - e^{2\pi ijx}}{e^{2\pi ijx} - 1}\end{aligned}$$
here $e^{2\pi ijx} - 1 \ne 0$ because $x \notin \mathbb{Q}$. This shows
$$\left|\frac{1}{N}\sum_{k=1}^{N} f(kx)\right| \le \frac{2}{N}\frac{1}{|e^{2\pi ijx} - 1|} = \frac{2}{N}\frac{1}{|\sin(\pi ijx)|} \to 0 = \int_0^1 f(x)dx.$$

2012Aug#7R. Consider the class of bounded operators
$$B(x,\cdot) : Y \to Z, \text{ where } x \in X, \|x\|_X \le 1.$$

Then for any $y \in Y$, $\{B(x,y) : x \in X, \|x\|_X \leq 1\}$ is bounded in Z. By the uniform boundedness principle, this implies

$$\|B(x,\cdot)\|_{Y\to Z} \leq C < \infty$$

for all x with $\|x\|_X \leq 1$. This implies

$$\|B(x,y)\|_Z \leq C\|x\|_X\|y\|_Y$$

for all $x \in X, y \in Y$.

2012Aug#8R. Notice that

$$\begin{aligned}\|f_n - f\|^2 &= \|f_n\|^2 + \|f\|^2 - \langle f_n, f\rangle - \langle f, f_n\rangle \\ &\to \|f\|^2 + \|f\|^2 - \langle f, f\rangle - \langle f, f\rangle \\ &= 0.\end{aligned}$$

2012Aug#9R. (a) Notice that if $\sum |c_n| < \infty$, then the distributional derivative (justify)

$$f' = \sum_{n=1}^{\infty} c_n \delta_{a_n} - \sum_{n=1}^{\infty} c_n \delta_{b_n}.$$

Hence

$$|\langle f', \varphi\rangle| \leq 2\big(\sum |c_n|\big) \max_{x\in\mathbb{R}} |\varphi(x)|.$$

This shows f' has order 0.

(b) If $\sum |c_n| = \infty$, then since c_n are reals we may assume that $\sum c_{n_k} = \infty$ where $c_{n_k} > 0$. Let $\chi_k \in C_c^\infty(b_{n_{k+1}}, b_{n_k})$ with $\chi(a_{n_k}) = 1$, and set

$$\varphi_N = \sum_{k=1}^{N} \chi_k.$$

Then $\varphi_N \in C_K^\infty(\mathbb{R})$ where $K = [0, b_1]$. However,

$$\langle f', \varphi_N\rangle = \sum_{k=1}^{N} \varphi_N(a_{n_k}) - \varphi_N(b_{n_k}) = N = N \max_{x\in\mathbb{R}} |\varphi_N(x)|.$$

This shows f' does not have order 0.

2012Aug#7C. Fix a branch of $\log z$ in $\mathbb{C}\backslash(-\infty, 0]$. By Cauchy's theorem,

$$\begin{aligned}\int_0^\infty \frac{\log x}{x^2-1}dx &= \int_0^\infty \frac{\log(xi)}{(xi)^2-1}d(xi) \\ &= -i\int_0^\infty \frac{\log x}{x^2+1}dx + \frac{\pi}{2}\int_0^\infty \frac{1}{x^2+1}dx \\ &= -i\int_0^\infty \frac{\log x}{x^2+1}dx + \left(\frac{\pi}{2}\right)^2.\end{aligned}$$

Thus the problem reduces to evaluating

$$\int_0^\infty \frac{\log x}{x^2+1}dx.$$

This integral is 0, because by change of variable $x = 1/y$.

$$\int_0^1 \frac{\log x}{x^2+1}dx = -\int_1^\infty \frac{\log y}{y^2+1}dy.$$

This can also be seen using contour integration: Fix a branch of $\log z$ in $\mathbb{C}\backslash[0,\infty)$ with $\log(ti) \in \mathbb{R} + \pi i$ for $t > 0$. For $\epsilon > 0$, consider the contour

$$\Gamma_\epsilon = \ell_+ \cup \gamma \cup \ell_-$$

oriented counter clockwise, where

$$\begin{aligned} \ell_+ &= \{t + i\epsilon : t \geq 0\} \\ \gamma &= \{z : |z| = \epsilon, \mathrm{Re} z \leq 0\} \\ \ell_- &= \{t - i\epsilon : t \geq 0\}. \end{aligned}$$

By the residue theorem

$$\begin{aligned} \int_{\Gamma_\epsilon} \frac{(\log z)^2}{z^2+1}dz &= -2\pi i\Big(\frac{(i\pi/2)^2}{2i} + \frac{(i3\pi/2)^2}{-2i}\Big) \\ &= -2\pi^3. \end{aligned}$$

On the other hand, as $\epsilon \to 0$, we have

$$\int_\gamma \frac{(\log z)^2}{z^2+1}dz \to 0$$

and

$$\begin{aligned} &\int_{\ell_+} \frac{(\log z)^2}{z^2+1}dz + \int_{\ell_-} \frac{(\log z)^2}{z^2+1}dz \\ &\to -\int_0^\infty \frac{(\log x)^2}{x^2+1}dx + \int_0^\infty \frac{(\log x + 2\pi i)^2}{x^2+1}dx \\ &= 4\pi i \int_0^\infty \frac{\log x}{x^2+1}dx - \int_0^\infty \frac{(2\pi)^2}{x^2+1}dx \\ &= 4\pi i \int_0^\infty \frac{\log x}{x^2+1}dx - 2\pi^3. \end{aligned}$$

Combining the above we get

$$\int_0^\infty \frac{\log x}{x^2+1}dx = 0$$

and hence

$$\int_0^\infty \frac{\log x}{x^2-1}dx = \frac{\pi^2}{4}.$$

2012Aug#8C. Assume that no such constant $c > 0$ exists. Then we can extract a sequence f_n satisfying the same conditions such that

$$f_n(z_0) \to 0, \text{ as } n \to \infty.$$

Since $|f_n(z)| \le 1$, by Montel's theorem (or the Arzelà-Ascoli theorem) there exists a subsequence n_k such that

$$f_{n_k} \to f$$

uniformly on $\mathbb{D}(0, 1-\epsilon)$. Where $\mathbb{D}(0, 1-\epsilon)$ is a disk centered at 0 containing z_0. By definition we have $f(z_0) = 0$, which contradicts Hurwitz's theorem since f_{n_k} has no zero.

2012Aug#9C. (a) Notice that for $y > 0$,

$$h(x+iy) - h(x-iy) = \int \frac{1}{y}\varphi\Big(\frac{t-x}{y}\Big)\chi_{(0,1)}(t)f(t)dt$$

where

$$\varphi(t) = \frac{1}{\pi}\frac{1}{t^2+1}.$$

Since $\frac{1}{y}\varphi\Big(\frac{t}{y}\Big)$ is an approximation to the identity and $\chi_{(0,1)}f$ is continuous at x, we conclude that, as $y \to 0^+$,

$$h(x+iy) - h(x-iy) \to f(x).$$

(b) Combining with (a), it suffices to show that

$$\lim_{y\to 0^+} h(x+iy) + h(x-iy)$$

exists. But

$$\begin{aligned}
&h(x+iy) + h(x-iy)\\
=&\frac{1}{\pi i}\int_0^1 \frac{(t-x)}{(t-x)^2+y^2}f(t)dt\\
=&\frac{1}{\pi i}\int_0^1 \frac{(t-x)}{(t-x)^2+y^2}\Big(f(t)-f(x)\Big)dt + \frac{1}{\pi i}\int_0^1 \frac{(t-x)}{(t-x)^2+y^2}f(x)dt\\
=&\frac{1}{\pi i}\int_0^1 \frac{(t-x)}{(t-x)^2+y^2}\Big(f(t)-f(x)\Big)dt + \frac{1}{\pi i}\int_{\substack{|t-x|>\delta\\ 0\le t\le 1}} \frac{(t-x)}{(t-x)^2+y^2}f(x)dt
\end{aligned}$$

where $\delta = \delta(x) > 0$ is fixed so that $(x-\delta, x+\delta) \subset (0,1)$. Since $f \in C^1$, the first integrand has bounded dominating function; since $|t-x| > \delta$, so does the second integrand. By the dominated convergence theorem, we conclude that

$$\begin{aligned}
&\lim_{y\to 0^+} h(x+iy)+h(x-iy)\\
&= \frac{1}{\pi i}\int_0^1 \frac{1}{t-x}\Big(f(t)-f(x)\Big)dt + \frac{1}{\pi i}\int_{\substack{|t-x|>\delta\\ 0\le t\le 1}} \frac{1}{t-x}f(x)dt.\\
&= \frac{1}{\pi i}\text{p.v.}\int_0^1 \frac{1}{t-x}f(t)dt.
\end{aligned}$$