Corrections are welcome.

2012Aug#1. By the fundamental theorem of calculus, we can write

$$f(x_1, x_2) = f(0, x_2) + \int_0^{x_1} f_1(s, x_2) ds.$$

For the same reason, we can further write

$$\begin{aligned} f(x_1, x_2) &= f(0, x_2) + \int_0^{x_1} f_1(s, 0) ds + \int_0^{x_1} \int_0^{x_2} g(s, t) dt ds \\ &= f(0, x_2) + \int_0^{x_1} f_1(s, 0) ds + \int_0^{x_2} \int_0^{x_1} g(s, t) ds dt. \end{aligned}$$

Now differentiating $f(x_1, x_2)$ with respect to x_2 gives (justify)

$$f_2(x_1, x_2) = f_2(0, x_2) + \int_0^{x_1} g(s, x_2) ds.$$

Differentiate $f_2(x_1, x_2)$ with respect to x_1, we get

$$\frac{\partial}{\partial x_1} f_2(x_1, x_2) = g(x_1, x_2),$$

as desired.

2012Aug#2. First, notice that the left hand side is quadratic in a and is minimized when

$$a = \frac{1}{|B_1|} \int_{B_1} f(y) dy.$$

(Note that picking $a = f(0)$ will not work for large n.) For $x \in B_1$ we write

$$\begin{aligned} |f(x) - a| &= \Big| \frac{1}{|B_1|} \int_{B_1} f(x) - f(y) dy \Big| \\ &= \Big| \frac{1}{|B_1|} \int_{B_1} \int_0^1 \nabla f(tx + (1-t)y) \cdot (x - y) dt dy \Big| \\ &\leq 2 \Big(\frac{1}{|B_1|} \int_{B_1} \int_0^1 |\nabla f(tx + (1-t)y)|^2 dt dy \Big)^{1/2}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{B_1} |f(x) - a|^2 dx &\leq \frac{4}{|B_1|} \int_{B_1} \int_{B_1} \int_0^1 |\nabla f(tx + (1-t)y)|^2 dt dy dx \\ &= \frac{4}{|B_1|} \int_0^{1/2} \int_{B_1} \int_{B_1} \cdots dy dx dt + \frac{4}{|B_1|} \int_{1/2}^1 \int_{B_1} \int_{B_1} \cdots dx dy dt \\ &=: I + II \end{aligned}$$

To estimate I, note that for $0 < t \le 1/2$,

$$\begin{aligned}\int_{B_1} |\nabla f(tx + (1-t)y)|^2 dy &= \frac{1}{(1-t)^n} \int_{B(tx,1-t)} |\nabla f(z)|^2 dz \\ &\le 2^n \int_{B_1} |\nabla f(z)|^2 dz.\end{aligned}$$

Hence

$$I \le 2^{n+1} \int_{B_1} |\nabla f(z)|^2 dz.$$

Similarly,

$$II \le 2^{n+1} \int_{B_1} |\nabla f(z)|^2 dz.$$

This shows one can take $C = 2^{n+2}$.

2012Aug#3. By conjugacy, the problem reduces to the case when $A = J_\lambda$ is a Jordan block, i.e. we need to find a complex matrix B so that

$$\exp B = J_\lambda.$$

Notice that

$$J_\lambda = \lambda(I + N)$$

where N is a nilpotent matrix. Formally, we have

$$\begin{aligned}\log(J_\lambda) &= \log\big(\lambda(I+N)\big) \\ &= \log\lambda + \log(I+N) \\ &= \log\lambda + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N^k.\end{aligned}$$

The last series is in fact a finite sum since N is nilpotent. Now if we take

$$B = (\log\lambda)I + \sum_{k\ge 1} \frac{(-1)^{k-1}}{k} N^k,$$

then it should be true that

$$\exp B = J_\lambda.$$

2012Aug#4. Consider the convolution,

$$f(x) = \chi_A * \chi_{-A}(x) = \int_{\mathbb{R}} \chi_A(x-y)\chi_{-A}(y)dy.$$

Then f is a continuous function since $\chi_A \in L^1$ and $\chi_{-A} \in L^\infty$. Moreover, f is supported on $A - A$, i.e. $f(x) \neq 0$ implies $x \in A - A$ (check). On the other hand,

$$f(0) = |A| \neq 0.$$

Hence $A - A$ contains an interval centered at 0.

2012Aug#5. First, notice that by Fatou's lemma (applied to $|f_n|^p$) we also have $\|f\|_p \le 1$.

By the Egorov's theorem, given $\epsilon > 0$, there exists $E \subset [0,1]$ such that $|[0,1]\backslash E| < \epsilon$ and f_n converges uniformly on E. Now

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^1 |f_n - f|^r dx \\
&\le \limsup_{n\to\infty} \int_E |f_n - f|^r dx + \limsup_{n\to\infty} \int_{[0,1]\backslash E} |f_n - f|^r dx \\
&= \limsup_{n\to\infty} \int_{[0,1]\backslash E} |f_n - f|^r dx.
\end{aligned}$$

By Hölder's inequality, for $q = p/r$ we have

$$\begin{aligned}
\int_{[0,1]\backslash E} |f_n - f|^r dx &\le \Big(\int_{[0,1]\backslash E} |f_n - f|^p dx \Big)^{1/q} |[0,1]\backslash E|^{1-1/q} \\
&\le \|f_n - f\|_p^{p/q} \epsilon^{1-1/q} \\
&\le 2^{p/q} \epsilon^{1-1/q}.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude

$$\limsup_{n\to\infty} \int_0^1 |f_n - f|^r dx = 0.$$

2012Aug#6. Notice that

$$\begin{aligned}
\sum_{n=-N}^{N} f_n &= \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(t) e^{-int} dt \\
&= \int_{-\pi}^{\pi} f(t) \big(\sum_{n=-N}^{N} e^{-int} \big) dt \\
&= \int_{-\pi}^{\pi} f(t) \frac{\sin((N+1/2)t)}{\sin(t/2)} dt \\
&= \int_{-\pi}^{\pi} \frac{f(t)}{\sin(t/2)} \sin((N+1/2)t) dt.
\end{aligned}$$

By the Riemann-Lebesgue lemma, to prove that

$$\sum_{n=-N}^{N} f_n \to 0$$

it suffices to show

$$\frac{f(t)}{\sin(t/2)} \in L^1[-\pi,\pi].$$

Since $|\sin(t/2)| \approx |t|$ when $t \in [-\pi, \pi]$ and $|f(t)| \le |\log|t||^{-2}$, it reduces to showing

$$\int_{-1/2}^{1/2} \frac{1}{|\log|t||^2} \frac{1}{|t|} dt < \infty.$$

But this is true since by a change of variable

$$\int_0^{1/2} \frac{1}{|\log|t||^2} \frac{1}{t} dt = \int_{\log 2}^{\infty} \frac{1}{s^2} ds < \infty.$$

2012Aug#7R. By considering instead $y_n = x_n - A$, we may assume $A = 0$. Now we need to find a sequence n_k so that

$$\Big\| \frac{1}{N} \sum_{k=1}^{N} x_{n_k} \Big\| \to 0, \text{ as } N \to \infty.$$

Notice that

$$\Big\| \frac{1}{N} \sum_{k=1}^{N} x_{n_k} \Big\|^2 = \frac{1}{N^2} \sum_{k=1}^{N} \|x_{n_k}\|^2 + \frac{1}{N^2} \sum_{\substack{i,j \le N \\ i \ne j}} \langle x_{n_i}, x_{n_j} \rangle.$$

Since the sequence x_n is bounded in $\mathcal{H}$, it suffices to make

$$\frac{1}{N^2} U_N \to 0$$

where

$$U_N := \sum_{\substack{i,j \le N \\ i \ne j}} \langle x_{n_i}, x_{n_j} \rangle.$$

Observe that

$$U_{N+1} - U_N = \langle x_{n_{N+1}}, x_{n_{N+1}} \rangle + \sum_{i \le N} \big(\langle x_{n_{N+1}}, x_{n_i} \rangle + \langle x_{n_i}, x_{n_{N+1}} \rangle \big).$$

Suppose $x_{n_i}, i = 1, 2, \cdots, N$ have been chosen. Since x_n converges weakly to 0, we can find n_{N+1} such that

$$\big| \langle x_{n_{N+1}}, x_{n_i} \rangle + \langle x_{n_i}, x_{n_{N+1}} \rangle \big| \le \frac{1}{N}.$$

This implies

$$|U_{N+1} - U_N| \le C$$

for some constant independent of N. In particular,

$$\big| \frac{1}{N^2} U_N \big| \le \frac{CN}{N^2} \to 0.$$

This finishes our inductive choice of n_k.

2012Aug#8R. By a smooth cutoff we can find $g \in C_c^{\infty}\big((-2,2) \times (-2,2)\big)$, such that $g = f$ on $[0,1] \times [0,1]$. Expand g into Fourier series, we get

$$g(x,y) = \sum_{(k,l) \in \mathbb{Z}^2} c_{k,l} e^{\frac{\pi}{2} i(kx+ly)}$$

where the Fourier coefficients $c_{k,l}$ decay rapidly in $|(k,l)|$. Notice that

$$e^{\frac{\pi}{2}i(kx+ly)} = e^{\frac{\pi}{2}ikx}e^{\frac{\pi}{2}ily}.$$

If we set

$$g_{k,l}(x) = c_{k,l}e^{\frac{\pi}{2}ikx}$$

and

$$h_{k,l}(x) = e^{\frac{\pi}{2}ily},$$

and reindex the sequence by j so that $\max(k_j, l_j)$ is nondecreasing in j. Then due to the rapid decay of c_j the conclusion follows.

2012Aug#9R. Define

$$\langle T, \phi \rangle = \int_{\mathbb{R}^n} \frac{1}{|x|^n}\big(\phi(x) - \phi(0)\big)dx.$$

Then T defines a distribution on $\mathbb{R}^n$ (check), and for any ϕ supported in $\mathbb{R}^n \backslash \{0\}$, we have

$$\langle T, \phi \rangle = \int_{\mathbb{R}^n} \frac{1}{|x|^n}\phi(x)dx.$$

2012Aug#7C. Identify $\mathbb{H}$ with $\mathbb{D}$. Then Schwarz lemma tells us that f is invariant on $\mathbb{D}(0,r)$. Pick $\epsilon > 0$ sufficiently small so that $ir \in \mathbb{D}(0, 1-\epsilon)$. Then by Cauchy's integral formula, we have

$$f'(ir) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}(0,1-\epsilon)} \frac{f(\zeta)}{(\zeta - ir)^2}d\zeta.$$

This implies

$$|f'(ir)| \le C_r \max_{\partial\mathbb{D}(0,1-\epsilon)} |f(\zeta)| \le C_r \max_{\mathbb{D}(0,1-\epsilon)} |\zeta|$$

which is independent of f.

2012Aug#8C. Identify $\mathbb{H}$ with $\mathbb{D}$. Then $F(z)$ is analytic on $\mathbb{D}$ and is continuous up to boundary by the dominated convergence theorem. By assumption there is a segment on $\partial\mathbb{D}$ on which $f \equiv C$. If we can show that this implies $f \equiv C$ on $\partial\mathbb{D}$, then by the Riemann-Lebesgue lemma we conclude $C = 0$.

To show that $f \equiv C$ on $\partial\mathbb{D}$, we may assume that $C = 0$, otherwise we can consider instead $F - C$. Now F vanishes on a segment on $\partial\mathbb{D}$. Choosing appropriate $\theta \in \mathbb{R}$ and N, we see that

$$G(z) = \prod_{k=1}^{N} F(e^{ik\theta}z)$$

vanishes on the whole boundary, and is analytic in $\mathbb{D}$. By the maximum modulus principle, this implies $g \equiv 0$ in $\mathbb{D}$ and hence $F \equiv 0$.

2012Aug#9C. Consider the contour

$$\Gamma_{R,\epsilon} = \gamma_0 \cup \ell_1 \cup \gamma_1 \cup \ell_2$$

oriented counter clockwise, where

$$\begin{aligned}
\gamma_0 &= \{z : |z| = R, \mathrm{Im} z \geq 0\} \\
\ell_1 &= \{t : -R \leq t \leq 1-\epsilon\} \\
\gamma_1 &= \{z : |z-1| = \epsilon, \mathrm{Im} z \geq 0\} \\
\ell_2 &= \{t : 1+\epsilon \leq t \leq R\}.
\end{aligned}$$

Write $f(z) = \frac{z}{z^3-1}$. By the residue theorem we have

$$\int_{\Gamma_{R,\epsilon}} f(z)dz = 2\pi i \mathrm{Res}(f, e^{i2\pi/3}) = -(e^{i2\pi/3}-1)^2 = -3e^{-\pi i/3}.$$

On the other hand, as $R \to \infty, \epsilon \to 0$,

$$\begin{aligned}
&\int_{\gamma_0} f(z)dz \to 0 \\
&\int_{\ell_0} f(z)dz + \int_{\ell_1} f(z)dz \to \lim_{x\to 0^+} \left\{ \int_{-\infty}^{1-\epsilon} + \int_{1+\epsilon}^{\infty} \right\} \frac{x}{x^3-1} dx \\
&\int_{\gamma_1} f(z)dz = \int_{\substack{|z-1|=\epsilon \\ \mathrm{Im} z \geq 0}} \frac{1}{z-1} \frac{z}{z^2+z+1} dz \to -\frac{\pi i}{3}.
\end{aligned}$$

Thus

$$\lim_{x\to 0^+} \left\{ \int_{-\infty}^{1-\epsilon} + \int_{1+\epsilon}^{\infty} \right\} \frac{x}{x^3-1} dx = \frac{\pi i}{3} - 3e^{-\pi i/3}.$$