Corrections are welcome.

2013Jan#1. Notice that

$$\frac{1}{x^x} = x^{-x} = e^{-x\log x} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x\log x)^n.$$

Integrating, we see that (justify the exchange of sum and integral)

$$\int_0^1 \frac{1}{x^x}dx = \sum_{n=0}^{\infty} \frac{1}{n!}\int_0^1 (-x\log x)^n dx.$$

Changing the variable $t = -\log x$, we have

$$\int_0^1 (-x\log x)^n dx = \int_0^{\infty} e^{-(n+1)t} t^n dt = \frac{1}{(n+1)^{n+1}} \int_0^{\infty} e^{-t} t^n dt.$$

Now integration by parts or noticing

$$\int_0^{\infty} e^{-t} t^n dt = \Gamma(n+1) = n!$$

gives the desired identity.

2013Jan#2. Notice that

$$f(x, y) = u \cdot (a, b, c)$$

where $u = \frac{1}{\sqrt{x^2+y^2+1}}(x, y, 1)$ is a unit vector ranging over the "northern hemisphere" not including the "equator". This inner product is largest when the angle between u and (a, b, c) and is smallest when the angle is largest. This visualization then quickly produces the answer.

2013Jan#3. It suffices to show

$$\limsup_{n\to\infty} \frac{x_n}{n} \le \liminf_{n\to\infty} \frac{x_n}{n} + \epsilon =: L + \epsilon$$

for all $\epsilon > 0$. Indeed, pick $m \ge 1$ such that

$$\frac{x_m}{m} < L + \epsilon.$$

Then for any n we can write $n = km + l$ where $0 \le l < m$ and $k = \lfloor n/m \rfloor$. By the condition we have

$$\frac{x_n}{n} \le \frac{kx_m}{n} + \frac{x_l}{n}$$

where we set $x_0 := 0$. Now notice that

$$x_l \le \max_{0\le j\le m} x_j =: M$$

and

$$k \le \frac{n}{m}.$$

Thus we have

$$\frac{x_n}{n} \leq \frac{x_m}{m} + \frac{M}{n}.$$

Letting $n \to \infty$, we see that

$$\limsup_{n\to\infty} \frac{x_n}{n} < L + \epsilon$$

which completes the proof.

2013Jan#4. One such example is given by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{1}{\sqrt{|x - q_n|}}$$

where $\{q_n\}_{n=1}^{\infty}$ is an enumeration of $\mathbb{Q} \cap [0, 1]$.

2013Jan#5. Notice that

$$\int_0^1 \sum_{n=1}^{\infty} |f(x-n)| dx = \int_{-\infty}^{0} |f(x)| dx < \infty$$

which shows that

$$\sum_{n=1}^{\infty} f(x-n)$$

converges a.e. on $[0, 1]$. The same argument applies to intervals $[k, k+1]$, $k \in \mathbb{Z}$ and hence shows the convergence on $\mathbb{R}$.

2013Jan#6. a) Notice that since $\left\{ \frac{1}{\sqrt{2\pi}} e^{ijx} \right\}_{j \in \mathbb{Z}}$ forms an orthonormal basis of $L^2[-2\pi, 2\pi]$, we have

$$S_n(f) := \sum_{j=-n}^{n} \left\langle f, \frac{e^{ij\cdot}}{\sqrt{2\pi}} \right\rangle \frac{e^{ijx}}{\sqrt{2\pi}} \to f \text{ in } L^2.$$

Hence in particular

$$\begin{aligned} \sum_{j=-n}^{n} c_j \int_a^b e^{ijx} dx &= \int_a^b S_n(f) dx \\ &= \left\langle \chi_{[a,b]}, S_n(f) \right\rangle \\ &\to \left\langle \chi_{[a,b]}, f \right\rangle \\ &= \int_a^b f(x) dx. \end{aligned}$$

This proves a).

b) Notice that $L^2[-2\pi, 2\pi] \subset L^1[-2\pi, 2\pi]$ is a dense subspace. Hence by the uniform boundedness principle , the limit

$$L_n(f) := \sum_{j=-n}^{n} c_j \int_a^b e^{ijx} dx \to L(f) := \int_a^b f(x) dx$$

holds also on $L^1[-2\pi, 2\pi]$ if and only if the functionals L_n satisfy

$$\|L_n\|_{(L^1)^*} \leq C$$

for some $C > 0$ independent of n. Direct computation shows that

$$L_n(f) = \langle \chi_{[a,b]} * D_n, f \rangle$$

where

$$D_n(x) = \sum_{j=-n}^{n} e^{ijx} = 1 + 2 \sum_{j=1}^{n} \cos(jx)$$

is the Dirichlet kernel. Hence the problem reduces to showing

$$\|L_n\|_{(L^1)^*} = \|\chi_{[a,b]} * D_n\|_{L^\infty} \leq C.$$

By the fundamental theorem of calculus, this reduces to the fact that

$$\sum_{j=1}^{n} \frac{\sin(jx)}{j}$$

is bounded uniformly in $x \in [0, 2\pi]$ and $n \geq 1$. But this is content of, for example, **2011Jan#3**.

2013Jan#7R. We will use the closed graph theorem. Suppose $x_n \to 0$ and $Ax_n \to z$, we need to show $z = 0$. Notice that by the condition

$$\langle Ax_n, y \rangle = \langle x_n, Ay \rangle$$

for any $y \in H$. Let $n \to \infty$, this gives

$$\langle z, y \rangle = \langle 0, Ay \rangle = 0.$$

Which implies $z = 0$.

2013Jan#8R. The condition implies

$$\left\langle T, \frac{\phi(\cdot - a) - \phi(\cdot)}{a} \right\rangle = 0.$$

Let $a \to 0$, we see that

$$\left\langle T, -\frac{d}{dx}\phi \right\rangle = 0,$$

or,

$$\frac{d}{dx} T = 0.$$

This implies $T = \text{const}$ by, e.g., **2008Aug#8R**.

2013Jan#9R. Notice that by definition, $f \in H^{\alpha}(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} (1+|\xi|^2)^{\alpha} |\hat{f}(\xi)|^2 d\xi < \infty.$$

Hence the problem reduces to examining the asymptotics of $\hat{f}(\xi)$ as $|\xi| \to \infty$.

If $f = f_2 = \delta'$, then

$$|\hat{f}(\xi)| \approx |\xi|.$$

Thus

$$\int_{\mathbb{R}} (1+|\xi|^2)^{\alpha} |\hat{f}(\xi)|^2 d\xi < \infty$$

if and only if $2\alpha + 2 < -1$, i.e. $\alpha < -\frac{3}{2}$.

If $f = f_1 = \chi_{|x|\le 1} \sin(x^3)$, then integration by parts shows

$$\hat{f}(\xi) = C\frac{\cos(2\pi\xi)}{\xi} + O\Big(\frac{1}{|\xi|^2}\Big), \text{ as } |\xi| \to \infty.$$

Thus

$$\int_{\mathbb{R}} (1+|\xi|^2)^{\alpha} |\hat{f}(\xi)|^2 d\xi < \infty$$

if and only if (check) $2\alpha - 2 < -1$, i.e. $\alpha < \frac{1}{2}$.

2013Jan#7C. Notice that the integrand is analytic on $\{z : |z| \ge 2\}$. Thus by Cauchy's theorem

$$\oint_{\gamma_2} \frac{z^5}{z^7 + 3z - 10} dz = \oint_{\gamma_R} \frac{z^5}{z^7 + 3z - 10} dz$$

for all $R > 2$. Let $R \to 2$ we see that the limit is 0.

2013Jan#8C. Let

$$g(z) = \frac{\frac{1}{2} - z}{1 - \frac{1}{2}z}.$$

Then $g \in \text{Aut}(\mathbb{D})$ and $g(0) = 1/2, g(1/2) = 0$. Consider the composition $F = f \circ g : \mathbb{D} \to \mathbb{D}$, which is satisfies $F(0) = 0$ and $F(1/2) = 2/5$. Assume for contradiction that $f'(1/2) = 0$, then by the chain rule $F'(0) = 0$. Apply the Schwarz lemma to $F(z)/z$, we conclude that $|F(z)| \le |z|^2$. But this contradicts $F(1/2) = 2/5$.

2013Jan#9C. (i) Notice that on $\text{Re}z \ge 1 + \epsilon$,

$$|(3k+5)^{-z}| \le (3k+5)^{-1-\epsilon}.$$

This implies uniform convergence of the series on $\text{Re}z \ge 1 + \epsilon$ and hence analyticity of $F(z)$ in $\text{Re}z > 1 + \epsilon$. Since ϵ is arbitrary, we get the conclusion.

(ii) First notice that for $\text{Re}z > 1$, by the fundamental theorem of calculus,

$$\int_1^{\infty} (3x+5)^{-z} dx = \frac{-1}{3(-z+1)} 8^{-z+1}.$$

Which is a meromorphic function with a pole at $z = 1$. Observe also that for Re$z \geq \epsilon$ and $k = 1, 2, \cdots$, by the mean value theorem,

$$\left|(3k+5)^{-z} - \int_k^{k+1} (3x+5)^{-z}dx\right| \leq 3|z|(3k+5)^{-1-\epsilon}.$$

Thus on any compact set in Re$z > 0$, the series

$$G(z) := \sum_{k=1}^{\infty}\Big((3k+5)^{-z} - \int_k^{k+1}(3x+5)^{-z}dx\Big)$$

converges uniformly, and so $G(z)$ defines an analytic function in Re$z > 0$. Now since on Re$z > 1$

$$F(z) = G(z) + \frac{1}{3(z-1)}8^{-z+1},$$

$F(z)$ can be continued to a meromorphic function in Re$z > 0$ by the above formula.

(iii) Since $G(z)$ is analytic in Re$z > 0$,

$$\begin{aligned}\int_\gamma F(z)dz &= \int_\gamma G(z)dz + \int_\gamma \frac{1}{3(z-1)}8^{-z+1}dz \\ &= 0 + \frac{1}{3}\int_\gamma \frac{1}{(z-1)}8^{-z+1}dz \\ &= \frac{2\pi}{3}i.\end{aligned}$$