1. Baire category

Basic tools: Baire category theorem.

2011Aug#7R. Assume that $f_n \in C([0,1])$ and, as a sequence of numbers, $f_n(x)$ converges to a finite number $f(x)$ for each $x \in [0,1]$. Prove that for any $\epsilon > 0$, there is a non-empty interval (a,b) and an integer p such that $\sup_{x\in(a,b)} |f(x) - f(y)| < \epsilon$ for any $n > p$.

2011Jan#7R. Is it possible to find a real-valued function f defined on $[0,1]$ such that $\lim_{x\to t} |f(x)| = \infty$ for every rational $t \in [0,1]$?

2010Aug#7R. Let $\{f_n\}_{n\geq 1}$ be a sequence of real valued continuous functions on the interval $[0,1]$, and let E be the set of $x \in [0,1]$ for which $\sup_n |f_n(x)| = \infty$.

Show that E cannot be $[0,1] \cap \mathbb{Q}$.

2009Aug#9R. If $I \subset \mathbb{R}$ is a closed interval, a function $f : I \to \mathbb{R}$ is *monotonic* on I if either it is non-decreasing on I or it is non-increasing on I. A function f on the interval $[0,1]$ is said to be *nowhere monotonic* if there is no closed subinterval $I \subset [0,1]$ on which f is monotonic. Prove that there exists a continuous function f on $[0,1]$ which is nowhere monotonic.
Hint: It may be easier to show more: the "typical" function in the space $C[0,1]$ of real-valued continuous functions on $[0,1]$ is nowhere monotonic.