1. Fourier series

Basic tools: Plancherel's theorem, Riemann-Lebesgue lemma, integration by parts, Dirichlet kernel, etc.

2013Jan#6. Assume that $f \in L^2[-\pi,\pi]$ and $c_j = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)e^{-ijx}dx$.
a) Prove that

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{j=-n}^{n} c_j \int_a^b e^{ijx}dx$$

for any $[a,b] \subset [-\pi,\pi]$.
b)* Does this statement remain true if $f \in L^1[-\pi,\pi]$?

2012Aug#6. Let f be a continuous function on $\mathbb{R}$ of period 1. Show that for each irrational x

$$N^{-1}\sum_{k=1}^{N} f(kx) \to \int_0^1 f(t)dt, \quad \text{as } N\to\infty.$$

Remark: This is called Weyl's equidistribution theorem.

2012Jan#6. Let $f \in L^1([-\pi,\pi])$ and $f_n = \int_{-\pi}^{\pi} f(t)e^{-int}dt$. Assume that

$$|f(t)| \le \frac{1}{\big|\log|t|\big|^2}.$$

Show that $\sum_{n=-N}^{N} f_n$ converges as $N\to\infty$.

2012Jan#8R.* Show that every function $f \in C^\infty([0,1]\times[0,1])$ can be written as

$$f(x,y) = \sum_{j=1}^{\infty} g_j(x)h_j(y), \quad x,y\in[0,1]$$

with $g_j, h_j \in C^\infty([0,1])$ satisfying, for every m,k,

$$\|g_j\|_{C^m([0,1])} \times \|h_j\|_{C^m([0,1])} \le \frac{C_{m,k}}{j^k}.$$

2011Aug#6.* Assume that $E\subset[0,2\pi]$ and its Lebesgue measure is positive.
(i) Show that, for any sequence t_n of real numbers,

$$\lim_{n\to\infty}\int_E \cos(nx+t_n)dx = 0.$$

(ii) Let a_n and b_n satisfy

$$\lim_{n\to\infty} a_n\cos(nx) + b_n\sin(nx) = 0, \quad \forall x\in E.$$

Prove that a_n and b_n tend to zero as $n \to \infty$.
Added: (iii) If in (ii) the limit exists and is finite, prove the same conclusion.

2010Aug#6. Let $I = [0,1]$, and define for $f \in L^2(I)$ the Fourier coefficients as

$$\widehat{f}_k = \int_0^1 f(t) e^{-2\pi k t} dt.$$

(i) Let $\mathcal{G}$ be the set of all $L^2(I)$ functions with the property that $|\widehat{f}_k| \leq |k|^{-3/5}$ for all $k \in \mathbb{Z}$. Prove that $\mathcal{G}$ is a compact subset of $L^2(I)$.

(ii) Let $\mathcal{E}$ be the set of all $L^2(I)$ functions with the property that $\sum_k |\widehat{f}_k|^{5/3} \leq 10^{-10}$. Is $\mathcal{E}$ a compact subset of $L^2(I)$?

2010Jan#5. 1) Let $f \in L^1([0,2\pi])$. Prove that $\int_0^{2\pi} f(x)\cos(nx)dx \to 0$, as $n \to \infty$. You are asked to a proof, not simply to quote a Theorem. By essentially the same proof, that you are not asked to repeat, one also has $\int_0^{2\pi} f(x)\sin(nx)dx \to 0$, as $n \to \infty$. Prove that, for any sequence (α_n) in $\mathbb{R}$, $\int_0^{2\pi} f(x)\cos^2(nx+\alpha_n)dx \to \frac{1}{2}\int_0^{2\pi} f(x)dx$.

2) Let (a_n) and (b_n) be sequences in $\mathbb{R}$ such that on a set of positive measure in $[0,2\pi]$, $a_n \cos nx + b_n \sin nx$ tends pointwise to 0. Prove that a_n and $b_n \to 0$.
Hint: Write $a_n \cos nx + b_n \sin nx = \rho_n \cos(nx + \alpha_n)$ *and use that fact that* $\cos^2\theta \leq |\cos\theta|$.