1. Lebesgue integration

Basic tools: Cauchy-Schwarz inequality, Hölder's inequality, Tonelli's theorem, Fubini's theorem, Minkowski inequality, etc.

2013Jan#4. Does there exist a function f such that $f \in L^1[0,1]$ but $\|f\|_{L^\infty[a,b]} = +\infty$ for any $[a,b] \subset [0,1], (a < b)$?

2012Aug#3. Let f be a C^1 function on $[0,1)$. Suppose that

$$\int_0^\infty t|f'(t)|^2 dt < \infty,$$

$$\lim_{T\to+\infty} T^{-1} \int_0^T f(t)dt = L.$$

Show that $f(t) \to L$ as $t \to +\infty$.

2012Jan#2. Let $B_r = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < r^2\}$. Let f be a C^1 real function on B_2. Prove that

$$\inf_{a\in\mathbb{R}} \int_{B_1} |f(x) - a|^2 dx \le C \sum_{j=1}^n \int_{B_1} \left|\frac{\partial f}{\partial x_j}\right|^2 dx$$

for a constant C independent of f.
Remark: This is called Poincaré inequality.

2012Jan#4.* Let $A \subset \mathbb{R}$ be a set of positive Lebesgue measure. Show that $A - A = \{x - y : x, y \in A\}$ contains a non-empty open interval.
Remark: This is called Steinhaus theorem.
Added: Let $A, B \subset \mathbb{R}$ be sets of positive Lebesgue measure. Show that $A + B = \{x + y : x \in A, y \in B\}$ contains a non-empty open interval.

2011Aug#2. Find all possible real numbers α, β such that

$$\int\int_{[0,1]\times[0,1]} \frac{1}{(x + y^\alpha)^\beta} dxdy < \infty.$$

2011Aug#4. Let f be a Lebesgue measurable function on $[0,1]$ with values in $\mathbb{R}$. Define

$$G(x,y) = f(x) - f(y)$$

and suppose that $G \in L^1([0,1] \times [0,1])$. Is $f \in L^1([0,1])$? Give a proof or find a counterexample.

2010Aug#4. For each part (i), (ii), (iii) either prove the statement or give a counterexample.

(i) Let P be a polynomial of two variables (not identically zero) and let $E = \{(x,y) \in \mathbb{R}^2 : P(x,y) = 0\}$.

Prove or disprove: E is a Lebesgue measurable set of measure zero.

(ii) Let $E_n \in [0,1]$ be Lebesgue measurable sets, with $\text{meas}(E_n) \leq n^{-1}$. Let $E = \limsup E_n$, i.e. the set of all $x \in [0,1]$ which belong to E_n for infinitely many n.

Prove or disprove: E is a Lebesgue measurable set of measure zero.

(iii) Let f and f_n belong to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, for every $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - f(x)|^2 dx = 0$.

Prove or disprove: $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx$.