1. Advanced calculus

Basic tools: Fundamental theorem of calculus, Stokes' theorem, mean value theorem, implicit function theorem, etc.

2013Jan#2. Find $\sup_{(x,y)\in\mathbb{R}^2} f(x,y)$ and $\inf_{(x,y)\in\mathbb{R}^2} f(x,y)$ if

$$f(x,y) = \frac{ax+by+c}{\sqrt{x^2+y^2+1}}, \quad a^2+b^2+c^2>0.$$

2012Jan#1. Let f be a continuous function on $\mathbb{R}^2$. Assume that the limit functions

$$\lim_{h\to 0}\frac{f(x_1+h,x_2)-f(x_1,x_2)}{h} = f_1(x_1,x_2)$$
$$\lim_{h\to 0}\frac{f(x_1,x_2+h)-f(x_1,x_2)}{h} = f_2(x_1,x_2)$$
$$\lim_{h\to 0}\frac{f_1(x_1,x_2+h)-f_1(x_1,x_2)}{h} = g(x_1,x_2)$$

exist and are continuous on $\mathbb{R}^2$. Prove that

$$\lim_{h\to 0}\frac{f_2(x_1+h,x_2)-f_2(x_1,x_2)}{h} = g(x_1,x_2).$$

Remark: Compare this with Clairaut's theorem.

2011Jan#2. Let $f:[a,b]\to\mathbb{R}$ be a continuous function and define

$$D^+f(x) = \limsup_{\substack{h\to 0\\ h>0}} \frac{f(x+h)-f(x)}{h}.$$

Show that if $D^+f(x)\geq 0$ for every $x\in[a,b]$ then $f(\alpha)\leq f(\beta)$ for $\alpha<\beta$ and $\alpha,\beta\in[a,b]$.
Hint: Consider $g(x)=f(x)+\epsilon x$ for small $\epsilon>0$.

2010Aug#1. Let $\mathcal{D}\subset\mathbb{R}^d, d\geq 2$ be a compact set with smooth boundary $\partial\mathcal{D}$ so that the origin belongs to the interior of $\mathcal{D}$. For every $y\in\partial\mathcal{D}$ let $\alpha(x)\in[0,\pi)$ be the angle between the position vector x and the outer normal vector $\mathfrak{n}(x)$. Let ω_d be the surface area of the unit sphere in $\mathbb{R}^d$. Compute

$$\frac{1}{\omega_d}\int_{\partial\mathcal{D}}\frac{\cos(\alpha(x))}{|x|^{d-1}}d\sigma(x)$$

where $d\sigma$ denotes surface measure on $\partial\mathcal{D}$.

Provide complete justifications for your computation. Does (a reasonable interpretation of) your result hold true if $d=1$?

2010Jan#1. 1) Evaluate

$$\int_\Gamma (x - y^3)dx + x^3 dy,$$

where Γ is the unit circle in $\mathbb{R}^2$, with counterclockwise orientation.

2) Find a function λ such that for any closed continuously differentiable curve $\mathcal{C}$ (i.e. a curve parameterized by a $\mathcal{C}^1$ map $t \to \gamma(t), t \in [0,1]$ with $\gamma(0) = \gamma(1)$).

$$\int_\Gamma (x - y^3)dx + x^3 dy = \int_{\mathcal{C}} \lambda(x,y)dy.$$

2010Jan#2. 1) Let f be a continuously differentiable function on $\mathbb{R}^2$. Assume that at each point

$$\frac{\partial f}{\partial x_1} > \Big|\frac{\partial f}{\partial x_2}\Big|.$$

Show that if $f(x_1, x_2) = f(x_1', x_2')$, then $|x_1' - x_1| < |x_2' - x_2|$ unless $(x_1, x_2) = (x_1', x_2')$.

2) Let ϕ be the map from $\mathbb{R}^2$ into itself defined by:

$$\phi(x_1, x_2) = \big(x_1 + \sin(\frac{x_1}{2} + \frac{x_2}{4}), x_2 + \sin(\frac{x_1}{4} + \frac{x_2}{2})\big).$$

Show that ϕ is a diffeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$, i.e. a 'smooth' map with smooth inverse. Evaluate the partial derivatives of ϕ^{-1} at the point $(0,1)$. *Hint. Question* 1 *can be used to prove injectivity. For proving surjectivity, a possibility is to prove that* $\phi(\mathbb{R}^2)$ *is closed and open in* $\mathbb{R}^2$.